A Generalised Theory of Proportionality in Collective Decision Making

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UNIVERSITAS VARSOVIENSIS

PRO-DEMOCRATIC

The model

1. A set of *candidates* or *projects* $C = \{c_1, c_2, \ldots, c_m\}$.
2. A set of voters $N = \{1, 2, \ldots, n\}$.

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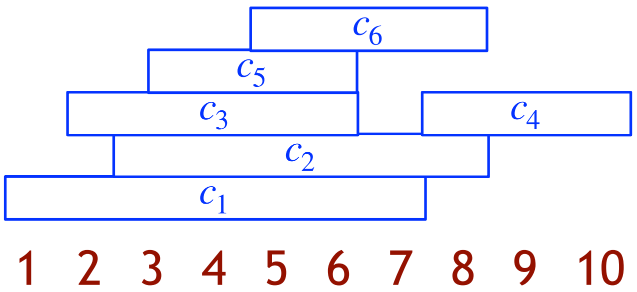

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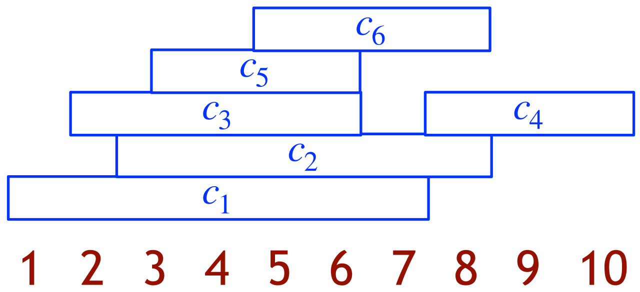

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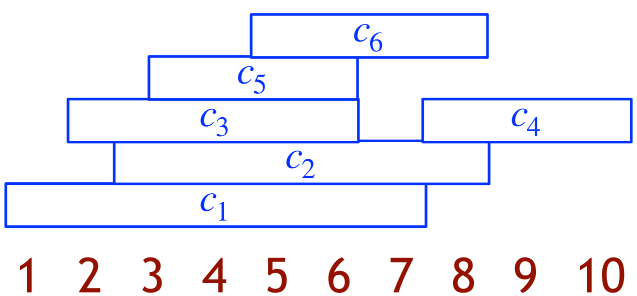
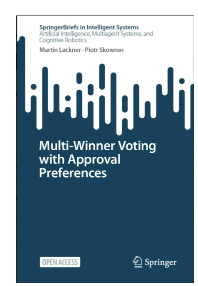


A subset of a given size k

$\downarrow$

committee elections



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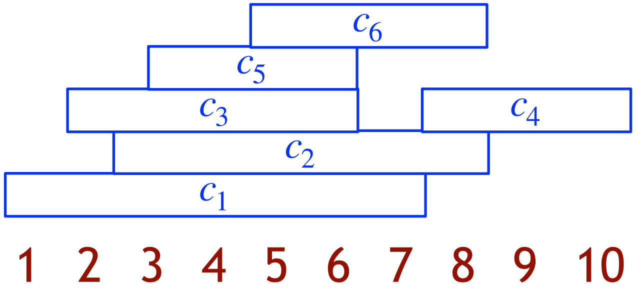

A subset of candidates with the total cost not exceeding the budget value b.

↓

participatory budgeting

S. Rey, J. Maly: The (Computational) Social Choice Take on Indivisible Participatory Budgeting, 2023.

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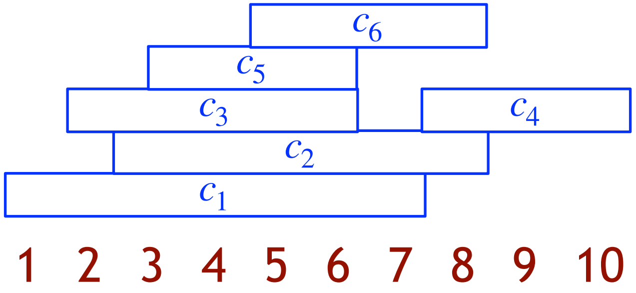

The candidates are grouped into pairs and foreach pair we need to select one.

public decisions

V. Conitzer, R. Freeman, and N. Shah. Fair public decision making. EC-2017.

R. Freeman, A. Kahng, and D. M. Pennock. Proportionality in approval-based elections with a variable number of winners. IJCAI-2020.

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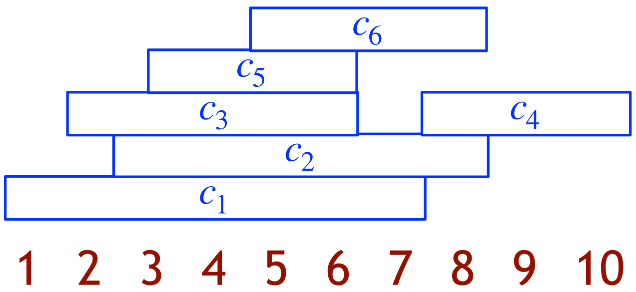

A subset of a given size k with **diversity constraints.**

L. E. Celis, L. Huang, and N. K. Vishnoi. Multiwinner voting with fairness constraints. IJCAI-2018.

R. Bredereck, P. Faliszewski, A. Igarashi, M. Lackner, and P. Skowron. Multiwinner elections with diversity constraints. AAAI-2018.

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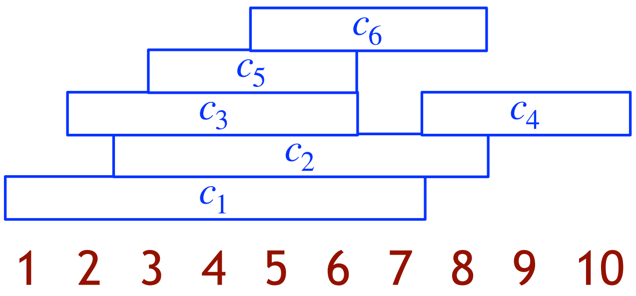

Ranking candidates

For each pair, c_1 and c_2, we introduce an auxiliary candidate $c_{1,2}$, whose selecting corresponds to ranking c_1 before c_2.

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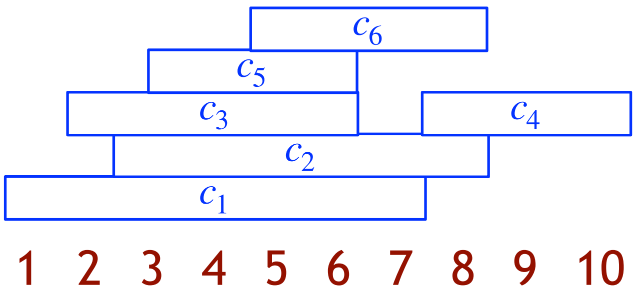

Committee elections with negative votes

For each c we introduce an auxiliary candidate $\bar{c}$, whose selecting corresponds to not selecting c.

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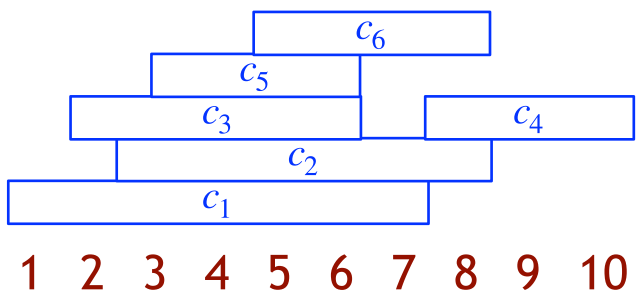

Our general model

We are given a nonempty **family of feasible sets** $\mathscr{F} \subseteq 2^C$.

($\mathscr{F}$ is closed under inclusions)

Proportionality in the general model

For committee elections:

An ℓ-cohesive group: a group of voters $S \subseteq N$ is cohesive if

(1) $|S| \geq \ell \cdot n/k$, and (2) $|\bigcap_{i \in S} A_i| \geq \ell$.

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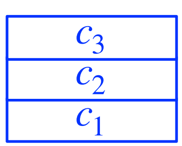
$k = 10$



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Extended Justified Representation (EJR): an outcome W satisfies extended justified representation if for each ℓ-cohesive group of voters S it holds that:

there exists $i \in S$ such that $|A_i \cap W| \geq \ell$

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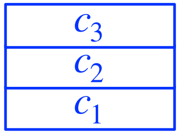

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The challange is how to properly define

ℓ-cohesiveness in the general model.

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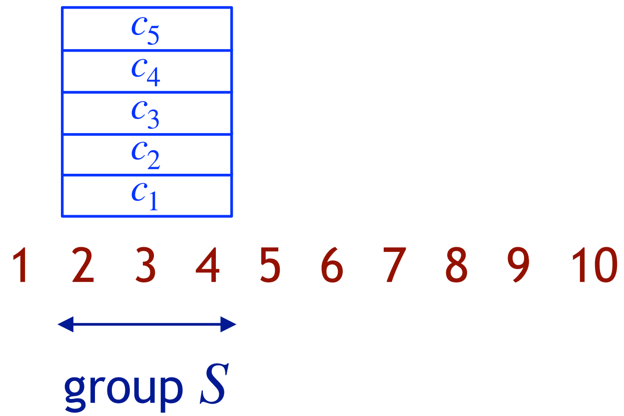

Group S agrees on some ℓ candidates.

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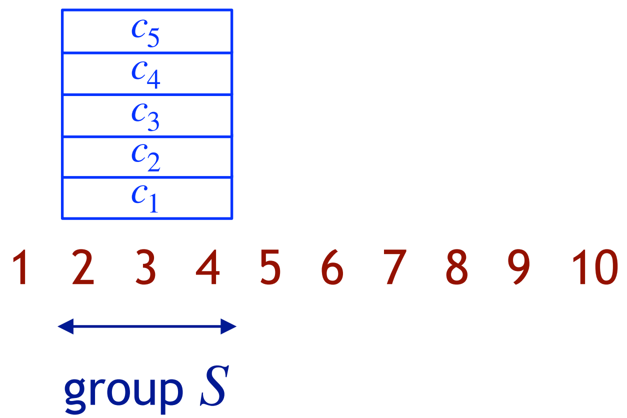

Group S agrees on some ℓ candidates. Do they deserve ℓ candidates?

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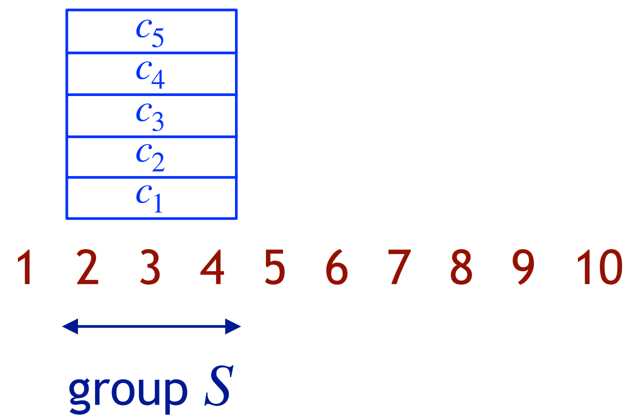

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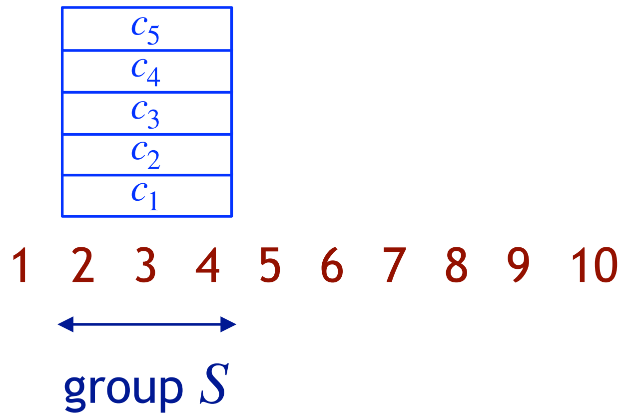

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$$\frac{\ell}{|S|} > \frac{|T|}{n - |S|}.$$

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2. If $|T| > 7$ then

$$\frac{|S|}{n} = 0.3 > \frac{3}{3 + |T|}$$

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Example (committee elections with 50% of men and 50% of women):

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The group S is entitled to 30% of 50 that is to 15 candidates.
(The hardest T consists of 35 woman.)

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This definition implies:

1. EJR in the model of committee elections.

H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified representation in approval-based committee voting. Social Choice and Welfare, 2017.

2. Strong EJR in the model of sequential decision making.

N. Chandak, S. Goel, and D. Peters. Proportional aggregation of preferences for sequential decision making. 2023.

3. Proportionality for cohesive groups in the model of public decisions.

P. Skowron and A. Górecki. Proportional public decisions. AAAI-2022.

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We can use this idea to extend other notions of propotionality.

$$\text{EJR} \downarrow \text{PJR}$$

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↓

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\text{proportionality degree of } \frac{\ell - 1}{2} & \text{PJR} \arrow[d] \arrow[r] & \text{lower quota} \\
 & \text{JR} &
\end{tikzcd}$$

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\end{tikzcd}$$

Why our definition is appealing?

1. It implies the stronges known JR-notions in the more specific models.

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2. **Theorem**: an outcome satisfying FJR always exists!

Why our definition is appealing?

3. **Theorem**: PAV satisfies EJR if and only if $\mathscr{F}$ is a matroid.

Proportional Approval Voting (PAV): select an outcome W that maximizes :

$$\sum_{i \in N} \mathrm{H}(|A_i \cap W|) \qquad \text{where} \qquad \mathrm{H}(z) = \sum_{j=1}^{z} \frac{1}{j}$$

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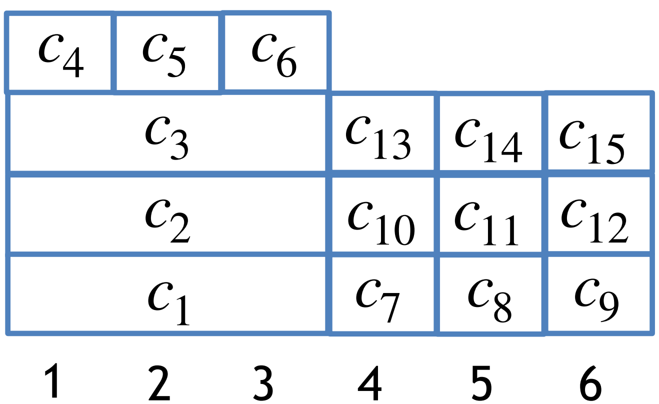
$$k = 12$$



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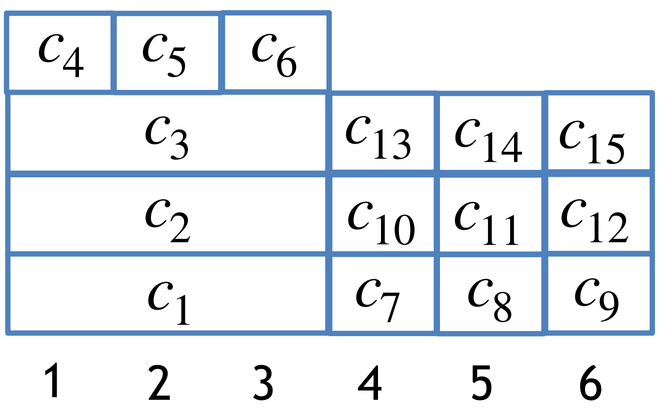

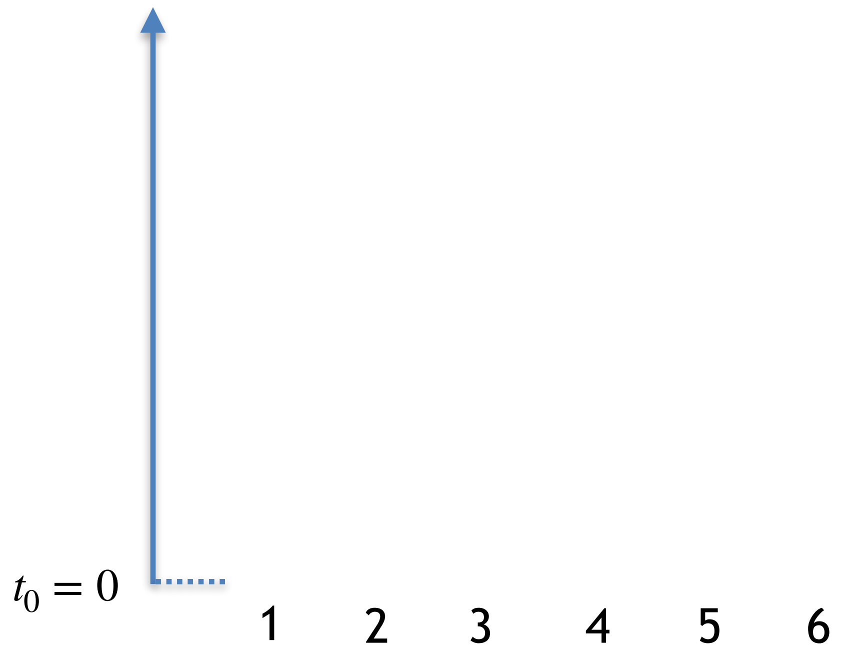

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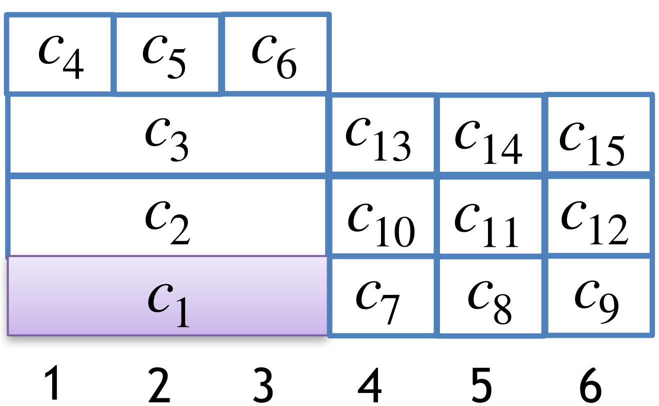

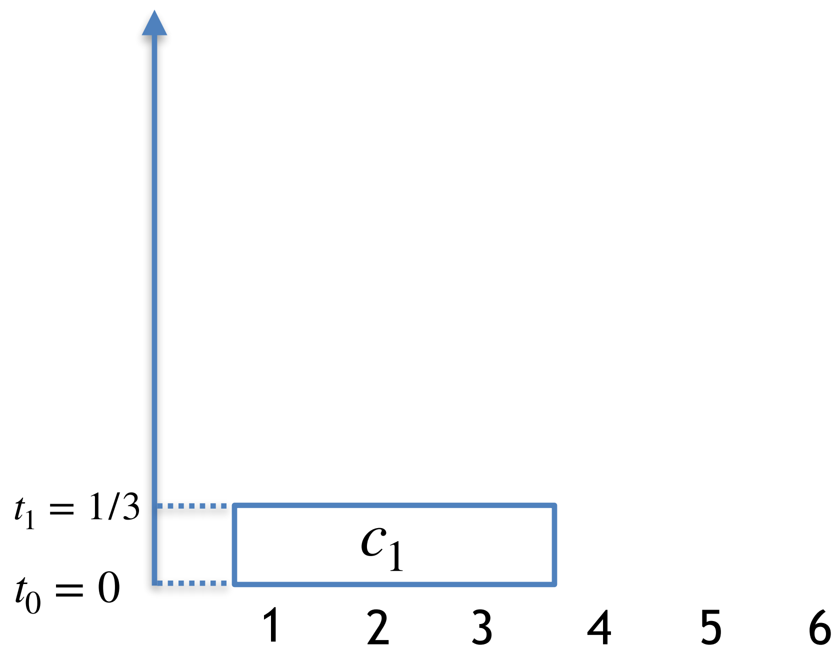

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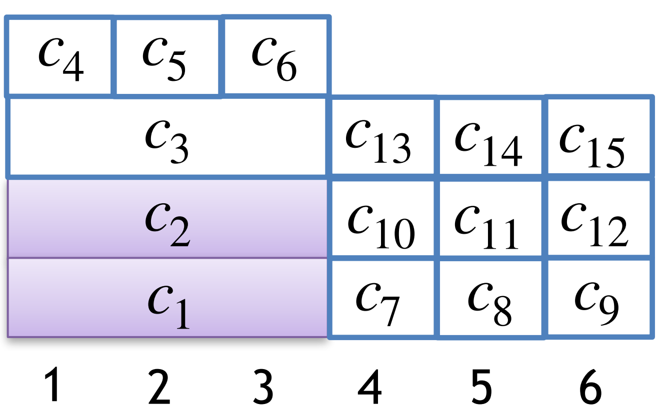

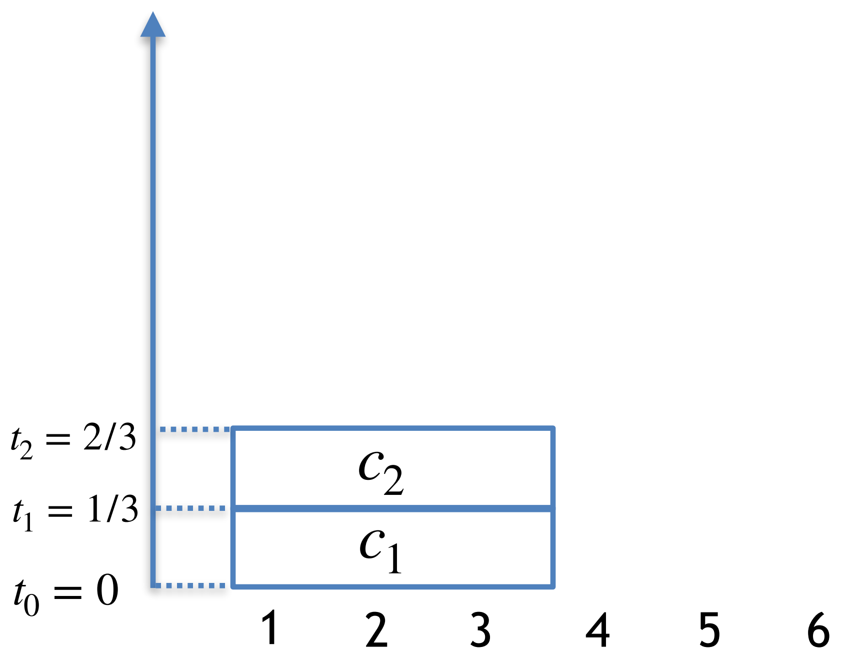

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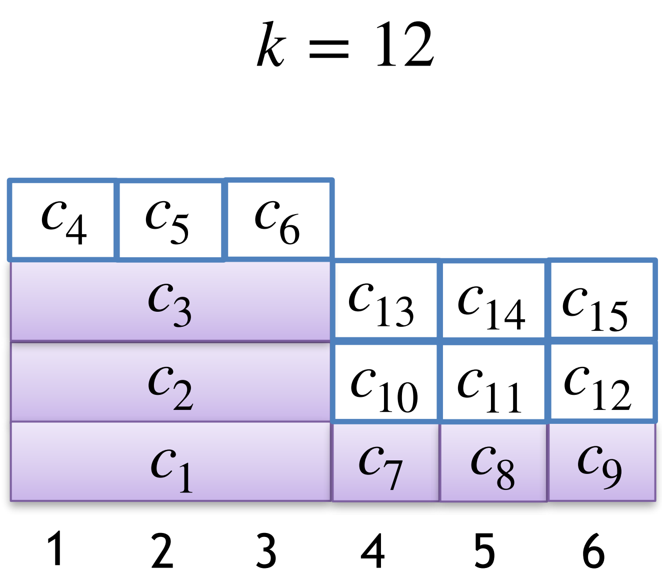

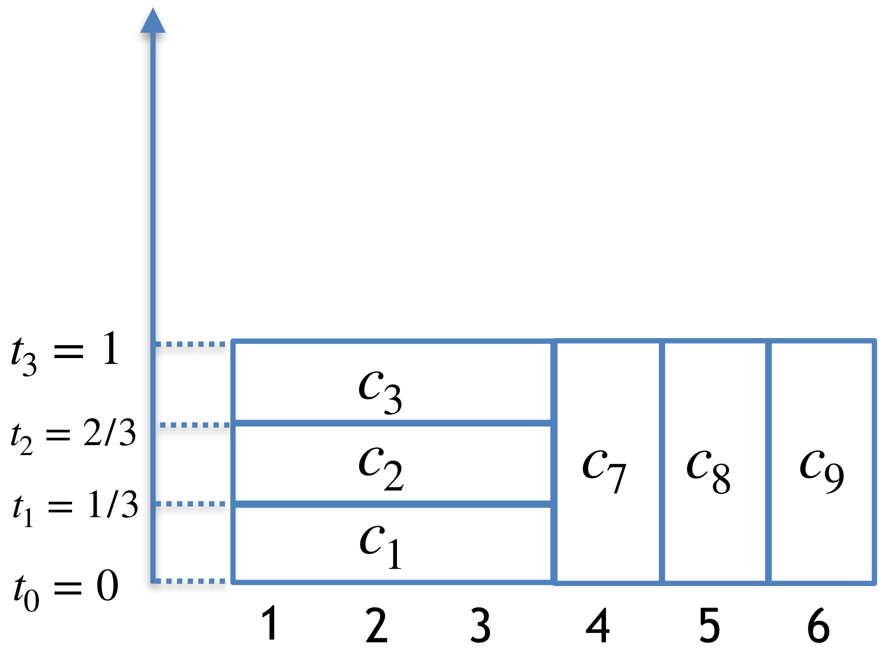

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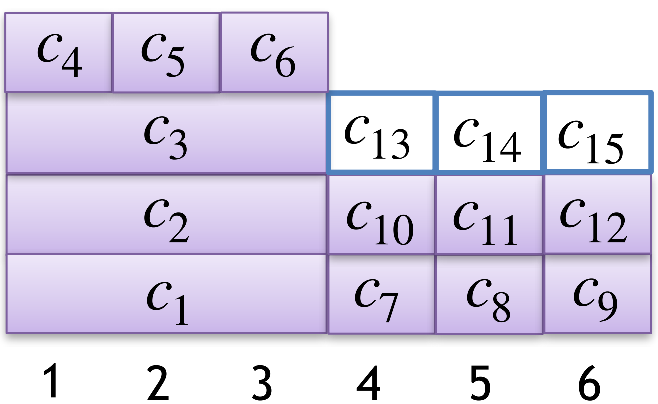

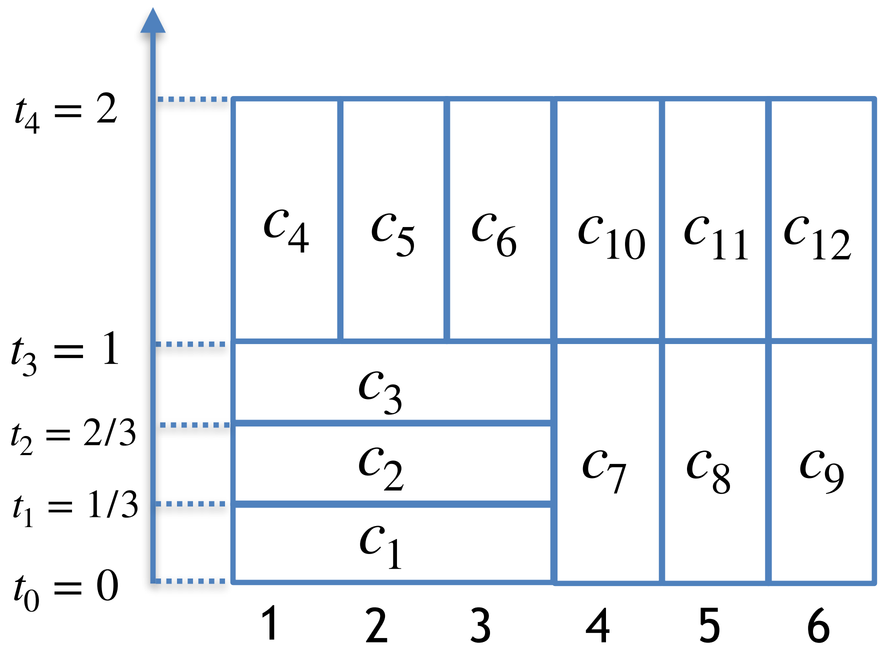

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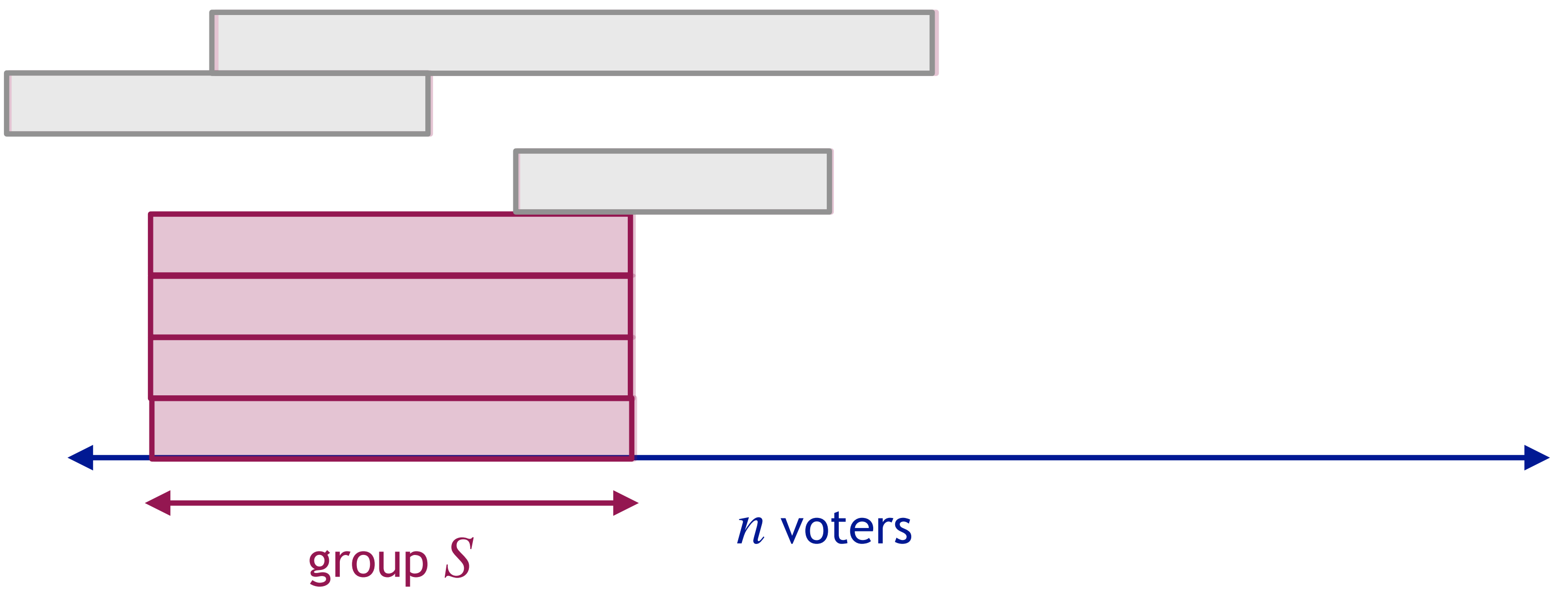

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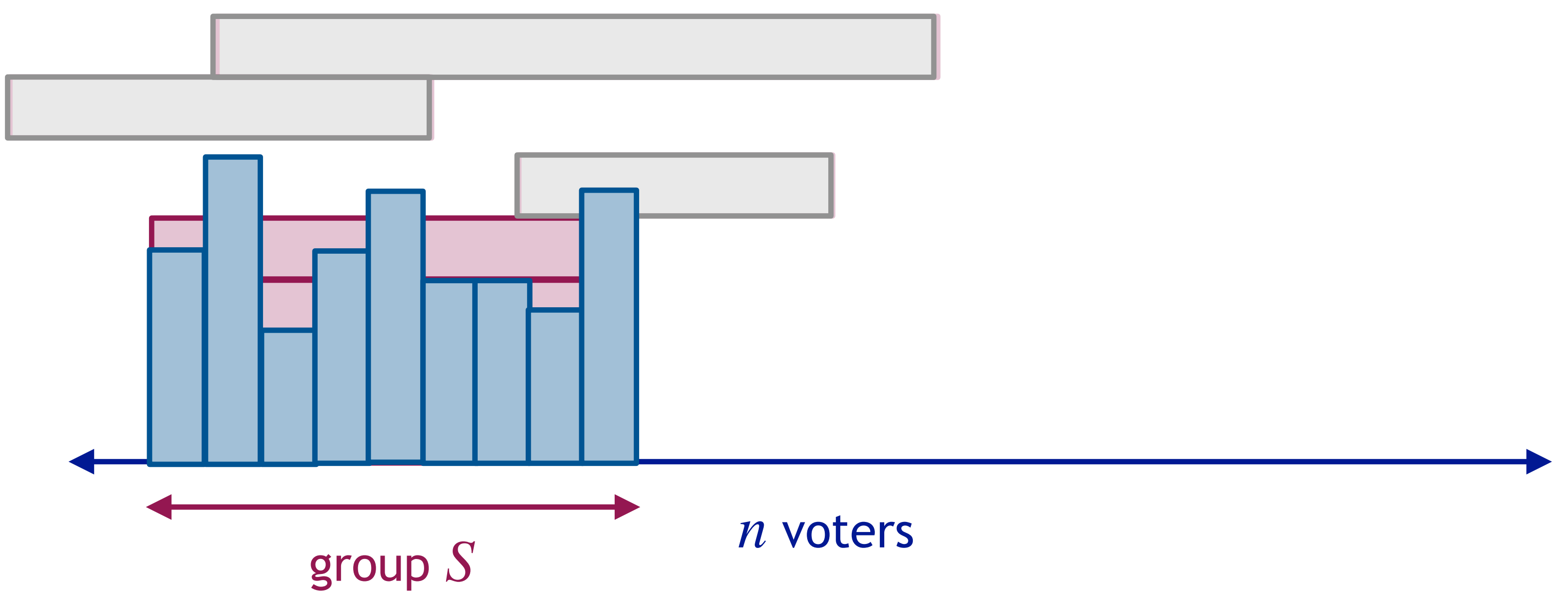

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D. Peters, G. Pierczyński, N. Shah, and P. Skowron. Market-based explanations of collective decisions. I AAAI-2021.

Why our definition is appealing?

1. It implies the stronges known JR-notions in the more specific models.
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2. Or

$$\frac{|S|}{n} > \frac{\ell}{|T| + \ell}$$

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When the candidates have weights

A group of voters $S \subseteq N$ is (α, β)-cohesive if for each feasible set $T \in \mathcal{F}$ at least one of the following conditions hold:

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Our results:

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