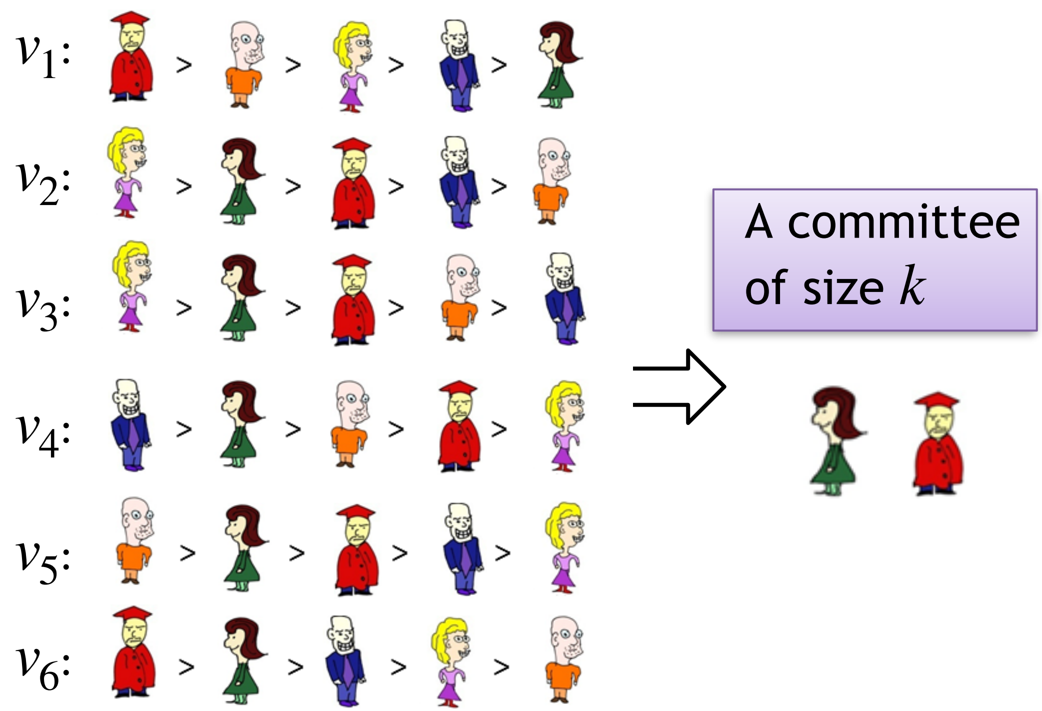
Proportional Algorithms:

Rankings

Piotr Skowron

University of Warsaw



Single Transferrable Vote (STV)

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1. If there exists a candidate ranked first by at least n/k voters, take this candidate to the committee, remove this candidate from the election, and remove some of her n/k supporters (voters who rank her first).

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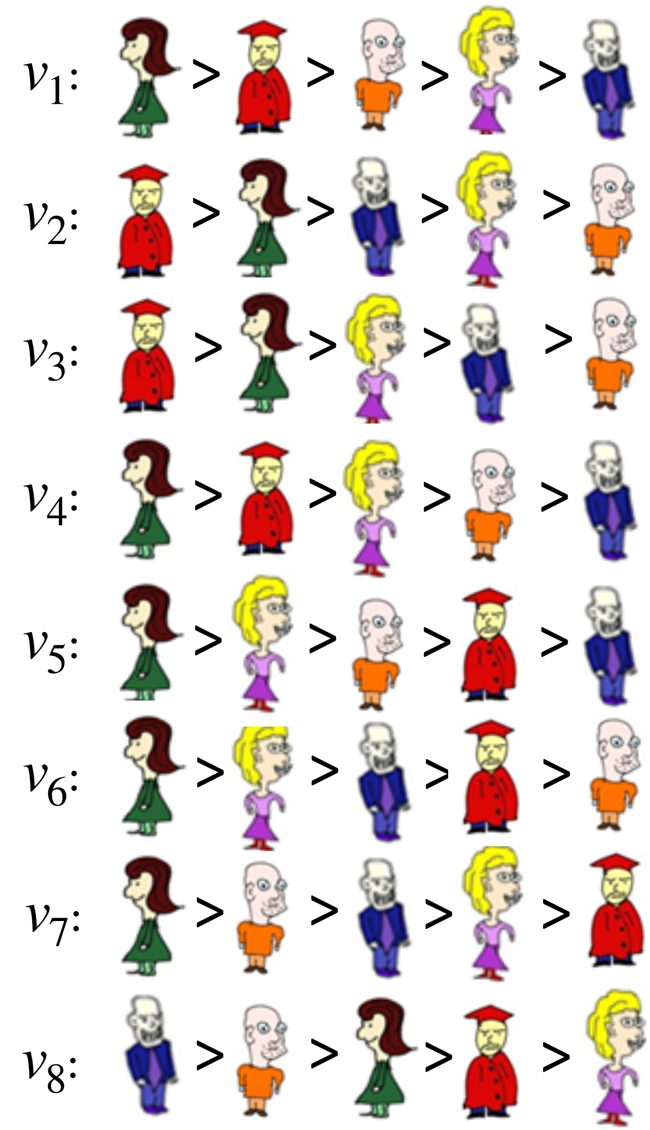

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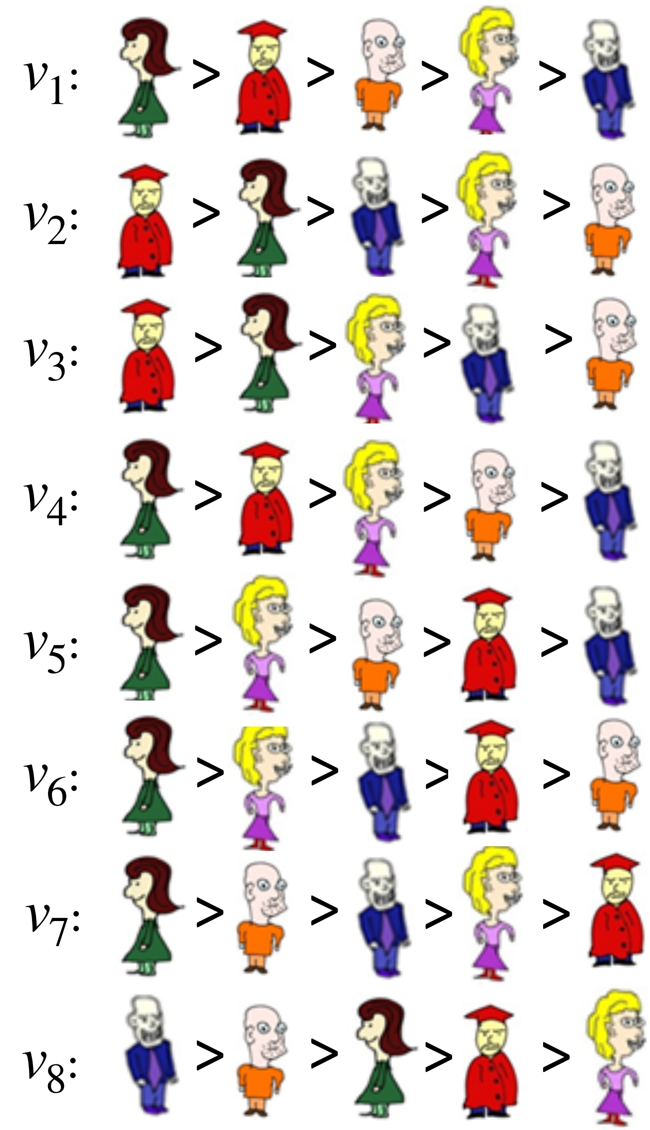

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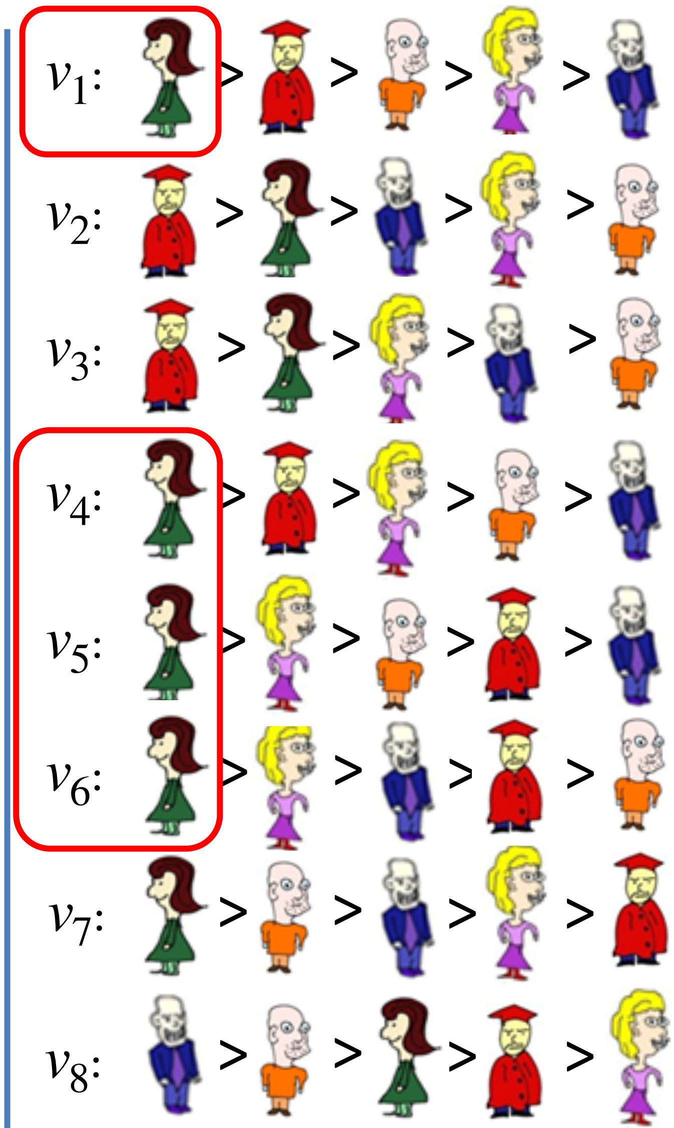

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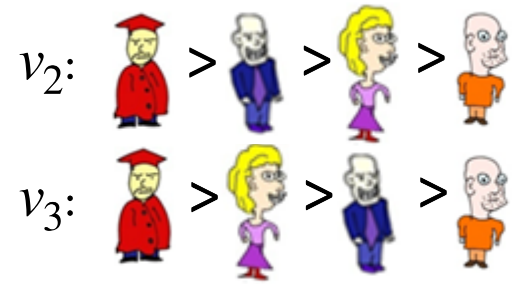

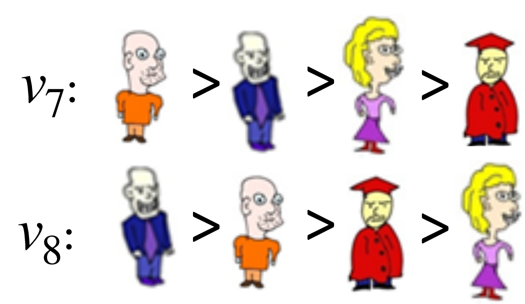

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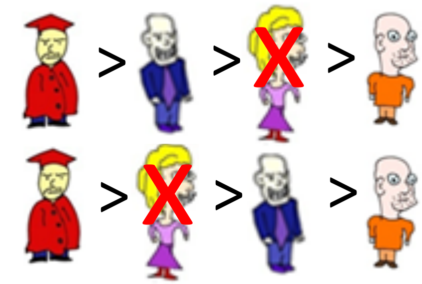
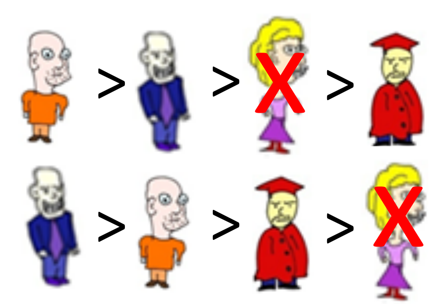


v_7:
v_8:



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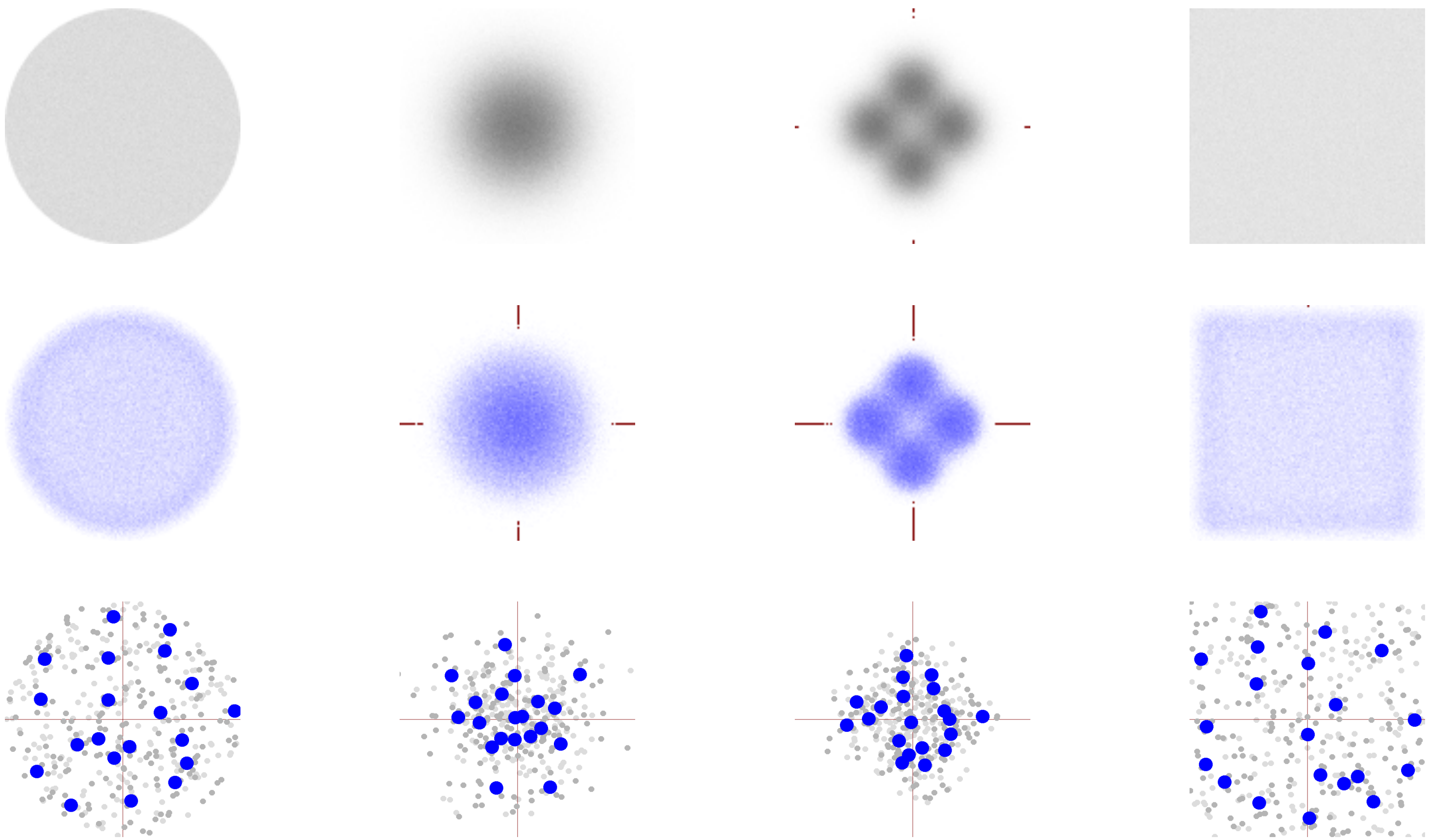
STV for 2-Euclidean Preferences

uniform on a circle	Gaussian	4 Gaussians	uniform on a square



STV and Proportionality for Solid Coalitions

Proportionality for Solid Coalitions (PSC). An outcome W satisfies PSC if for each $\ell \in [k]$, each subset of voters $S \subseteq N$ with $|S| \geq n\ell/k$ and each subset of candidates T such that $T \succ_i C \setminus T$ for all $i \in S$, it holds that:

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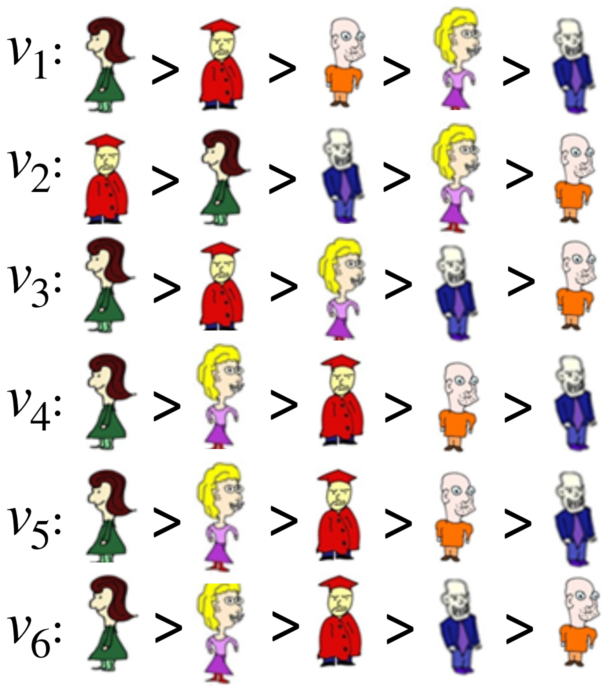

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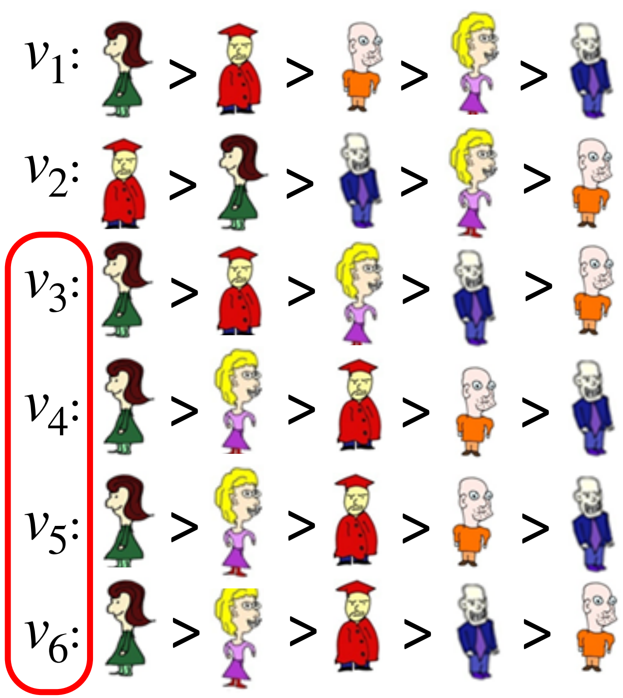

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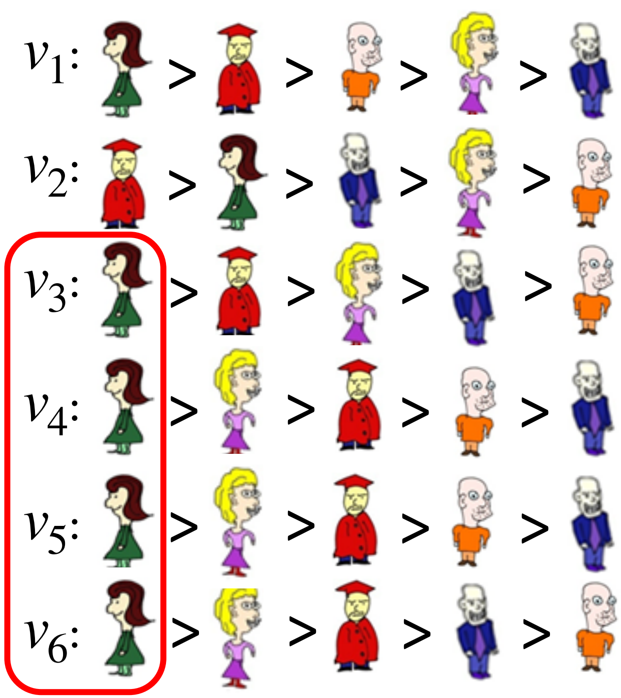

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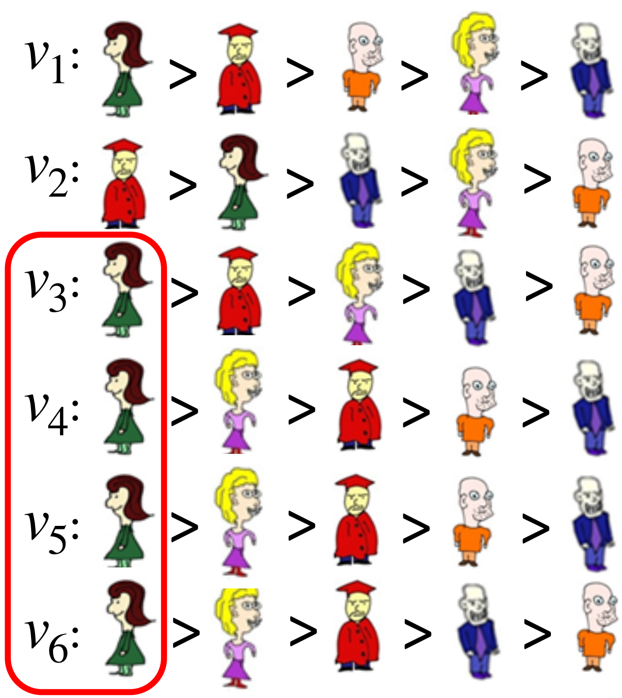

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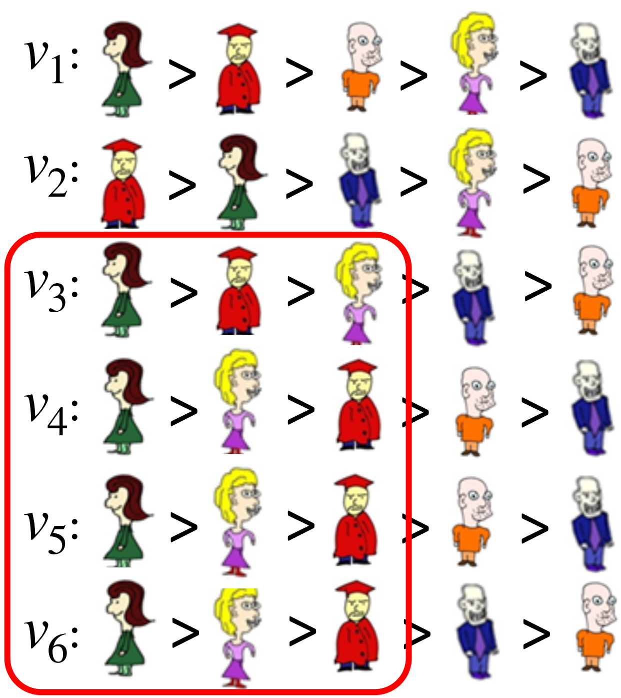

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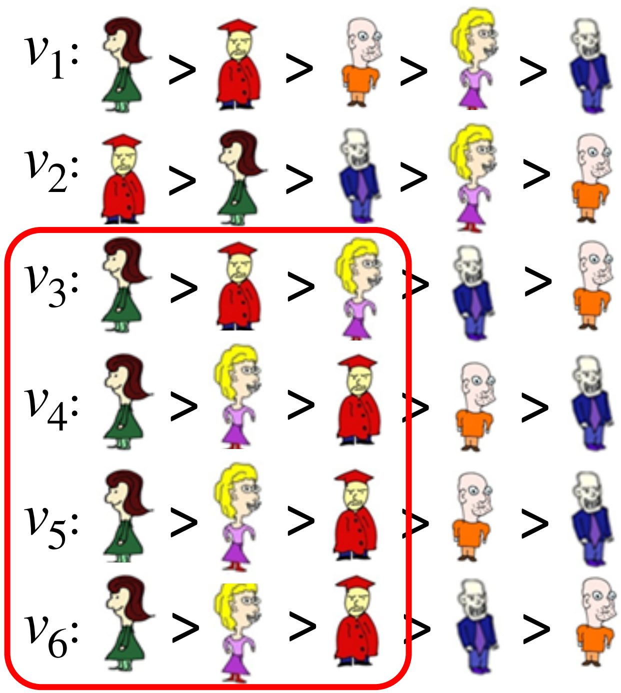

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If there is $c \in T$ ranked first by at least n/k voters, then take c to the committee, and remove n/k voters from S. The new group S' and the new set T' will satisfy premises and so, by induction hypothesis:

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If we remove candidate $c \in T$ then still T' contains at least ℓ candidates (and so, the thesis for T' will imply the thesis for T).

STV and monotonicity

Consider STV in this example:

- Candidate A will be eliminated first.

5					6						4				2	
v1	v2	v3	v4	v5	v6	v7	v8	v9	v10	v11	v12	v13	v14	v15	v16	v17
A	A	A	A	A	C	C	C	C	C	C	B	B	B	B	B	B
C	C	C	C	C	B	B	B	B	B	B	A	A	A	A	C	C
B	B	B	B	B	A	A	A	A	A	A	C	C	C	C	A	A

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					C	C	C	C	C	C	B	B	B	B	B	B
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Consider STV in this example:

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Monotonicity: if a voter pushes a winning candidate up in her ranking, then this candidate should still be winning.

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STV is non-monotonic!

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STV and monotonicity

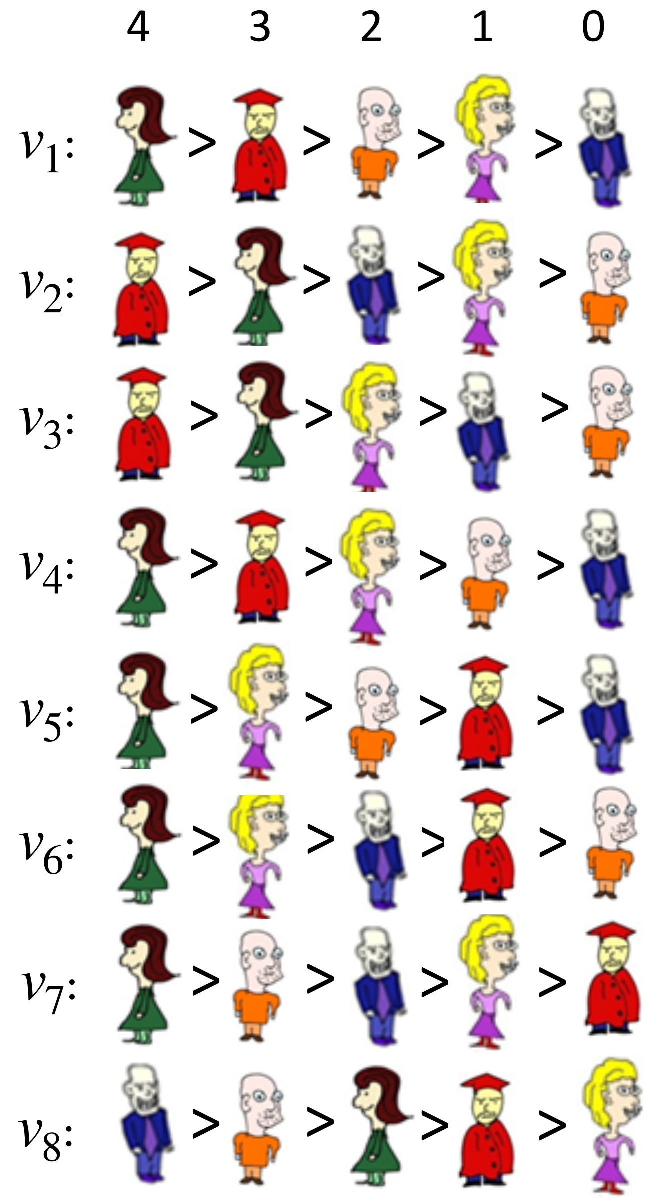
Open question: Is there a rule that satisfies proportionality for solid coalitions and monotonicity?

The Monroe rule

Define the score for a committee: 

Find the **best** assignment of voters to committee members so that:

Each committee member is assigned to roughly n/k voters.



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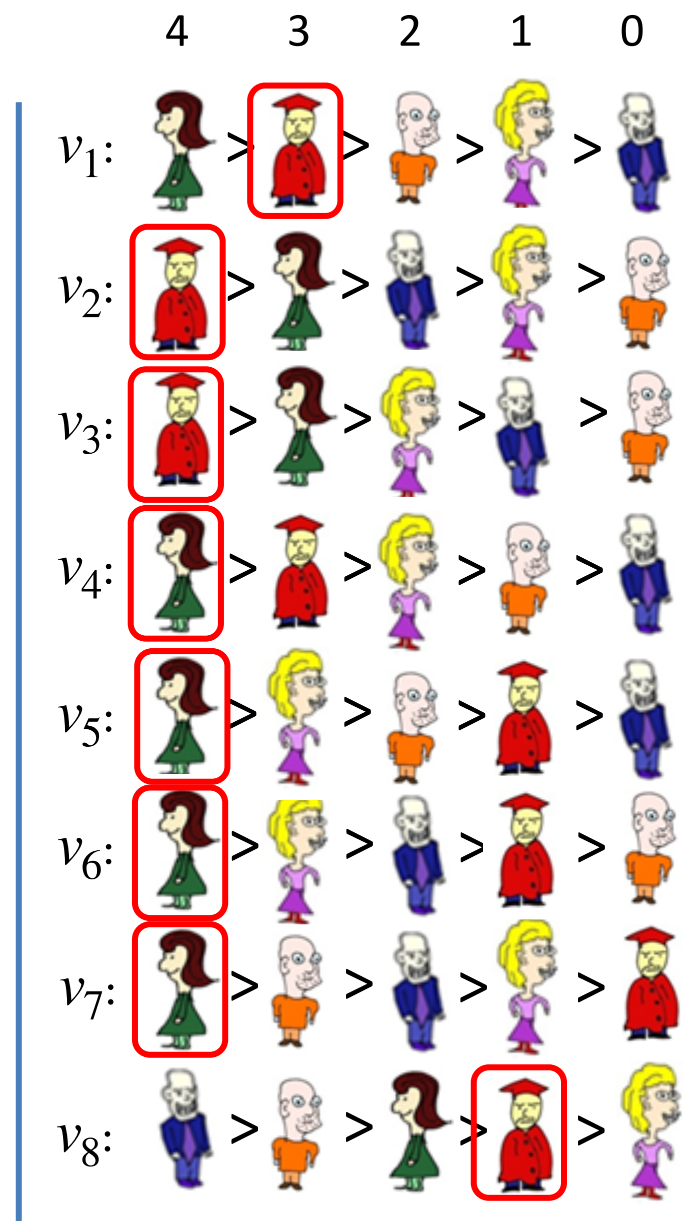

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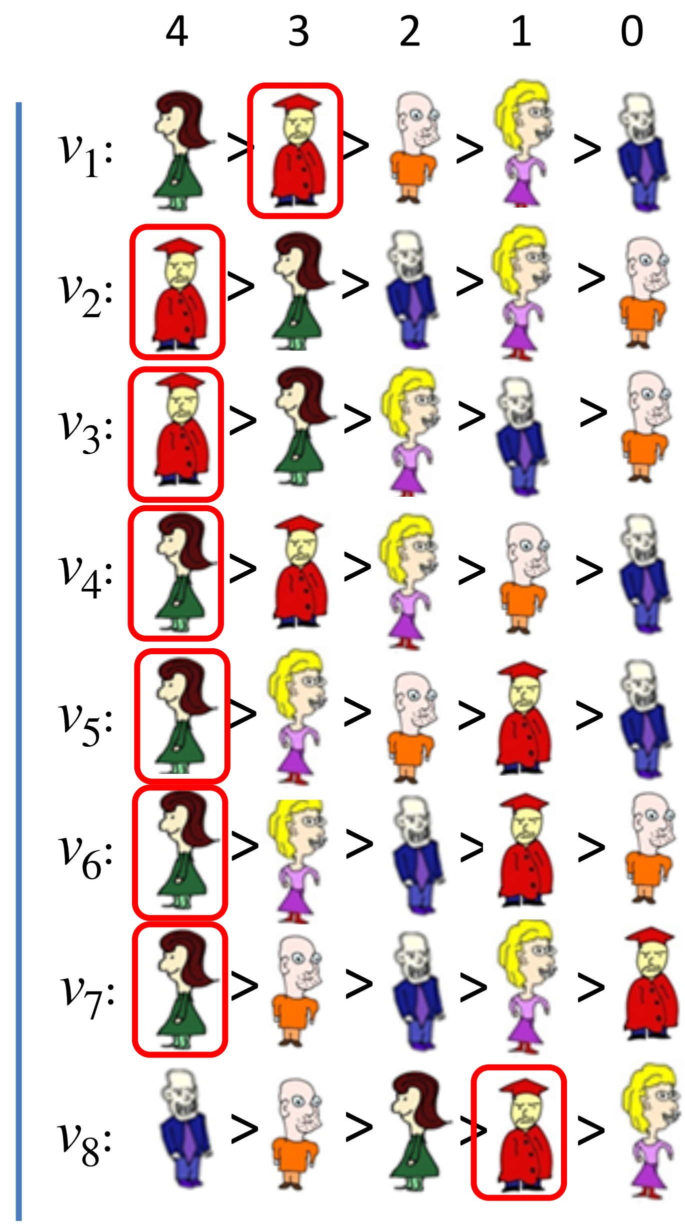
This assignment has score:

$3 + 6 \cdot 4 + 1 = 28$



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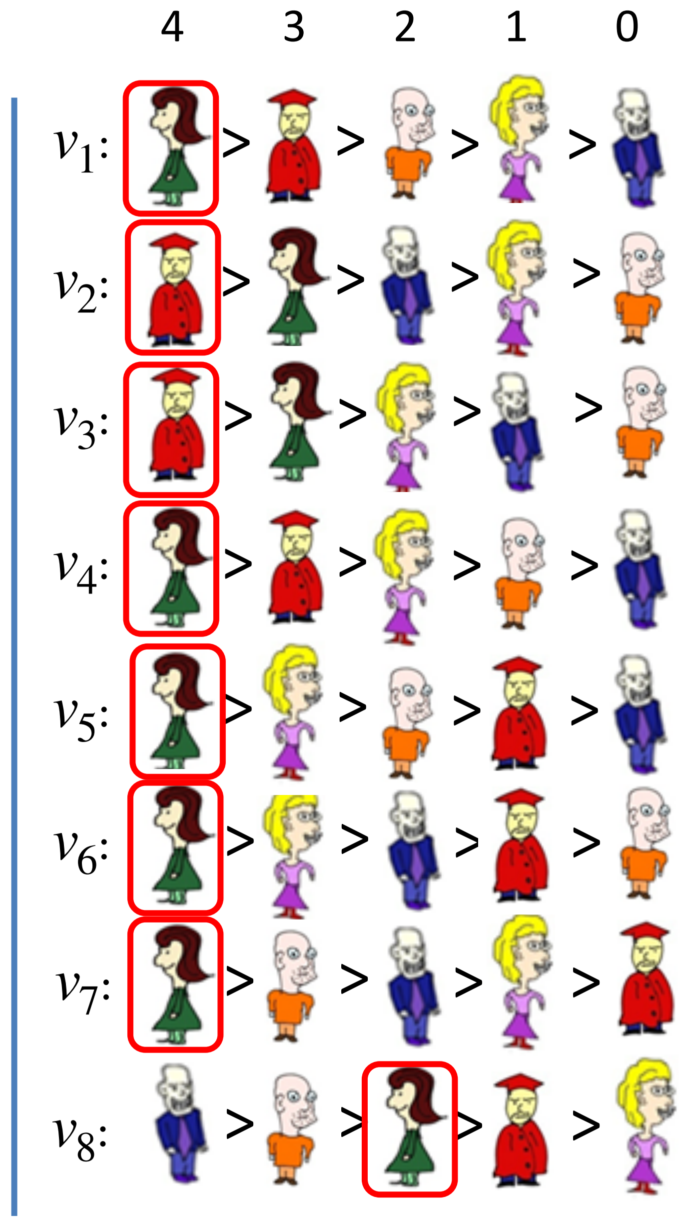

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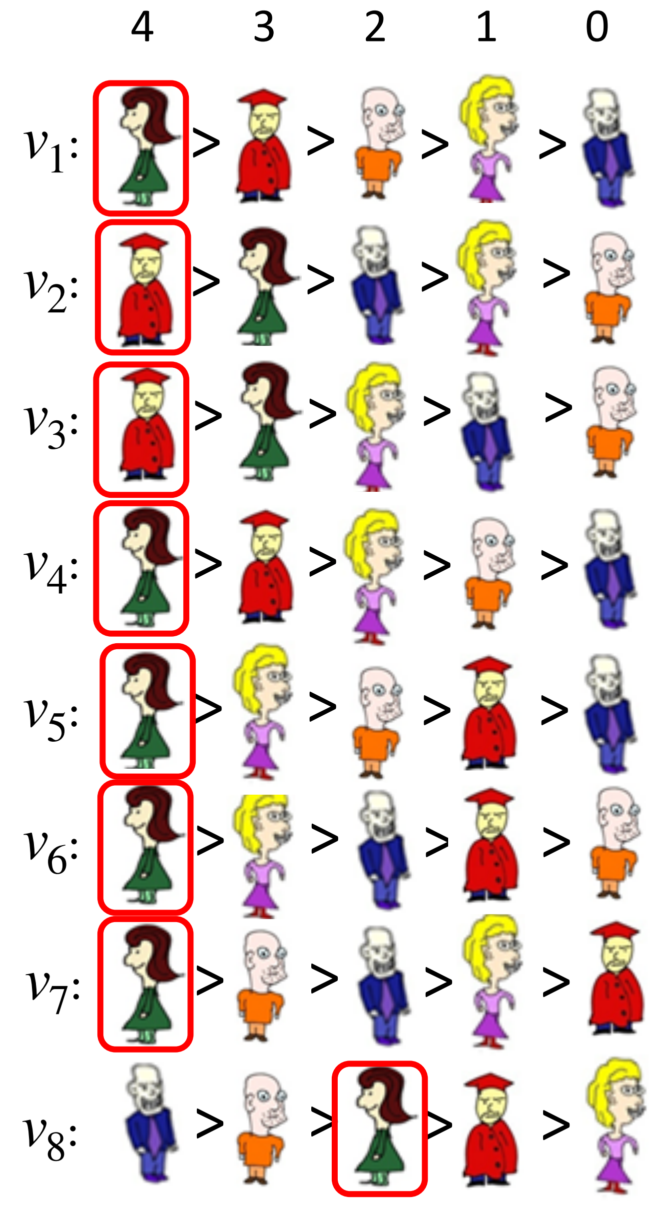
This would be a better assignment with score of 30.



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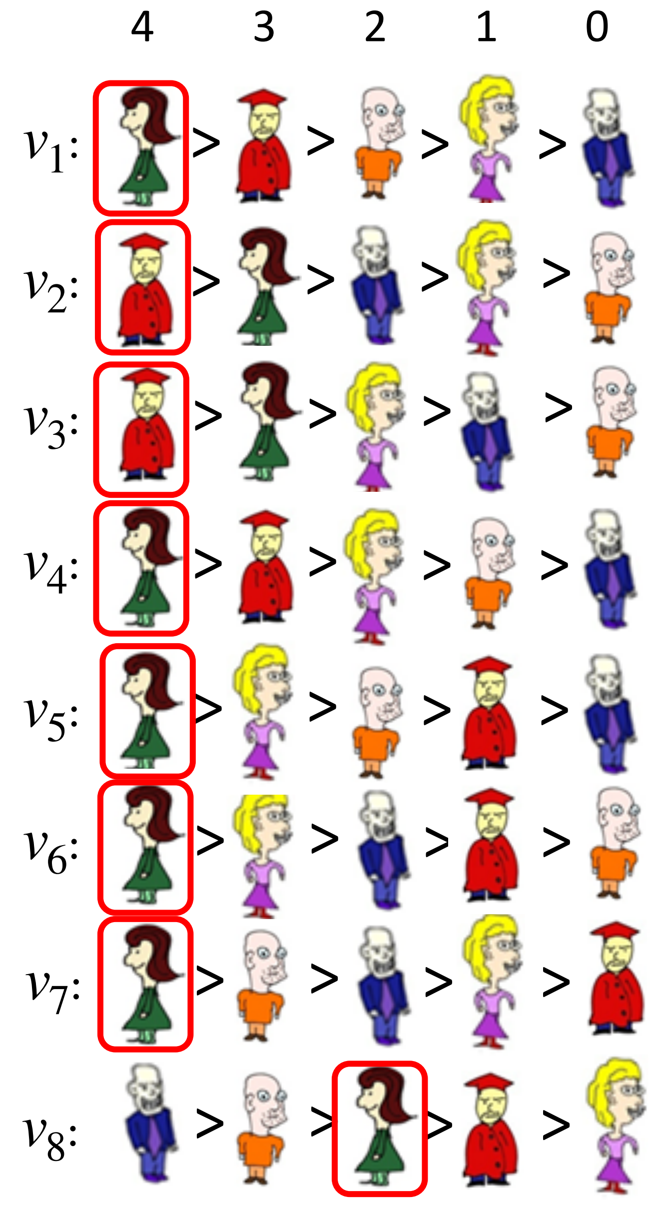
But this assignment is unbalanced and so it is not valid!



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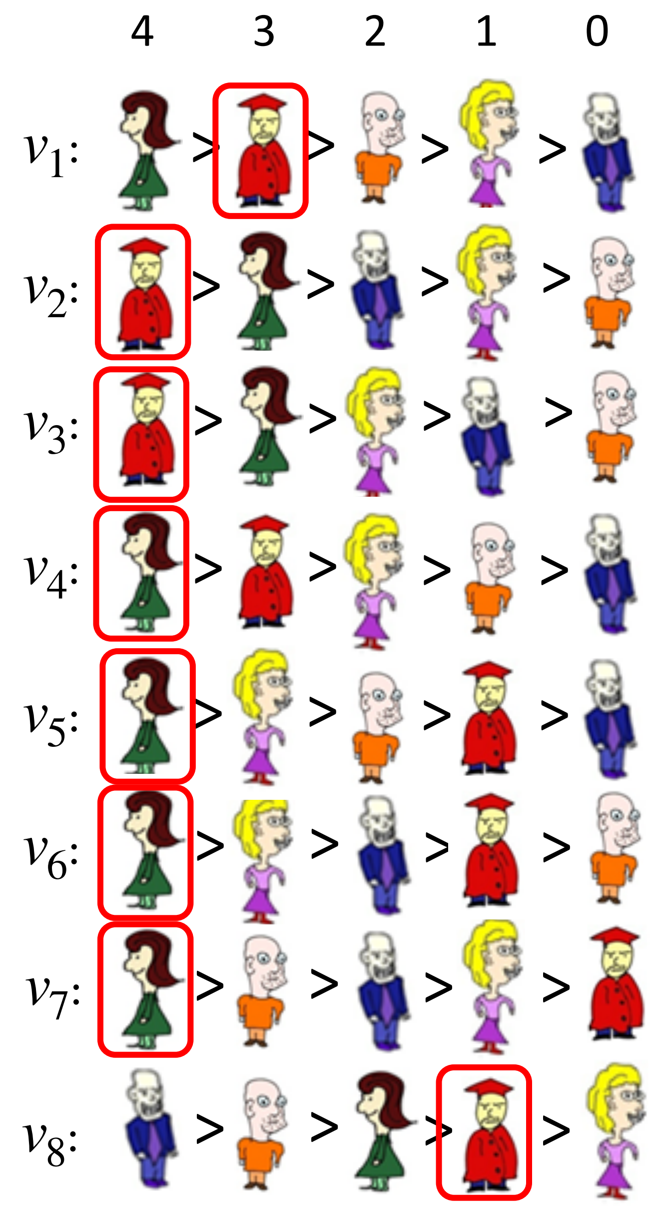
Each committee member is assigned to roughly n/k voters.

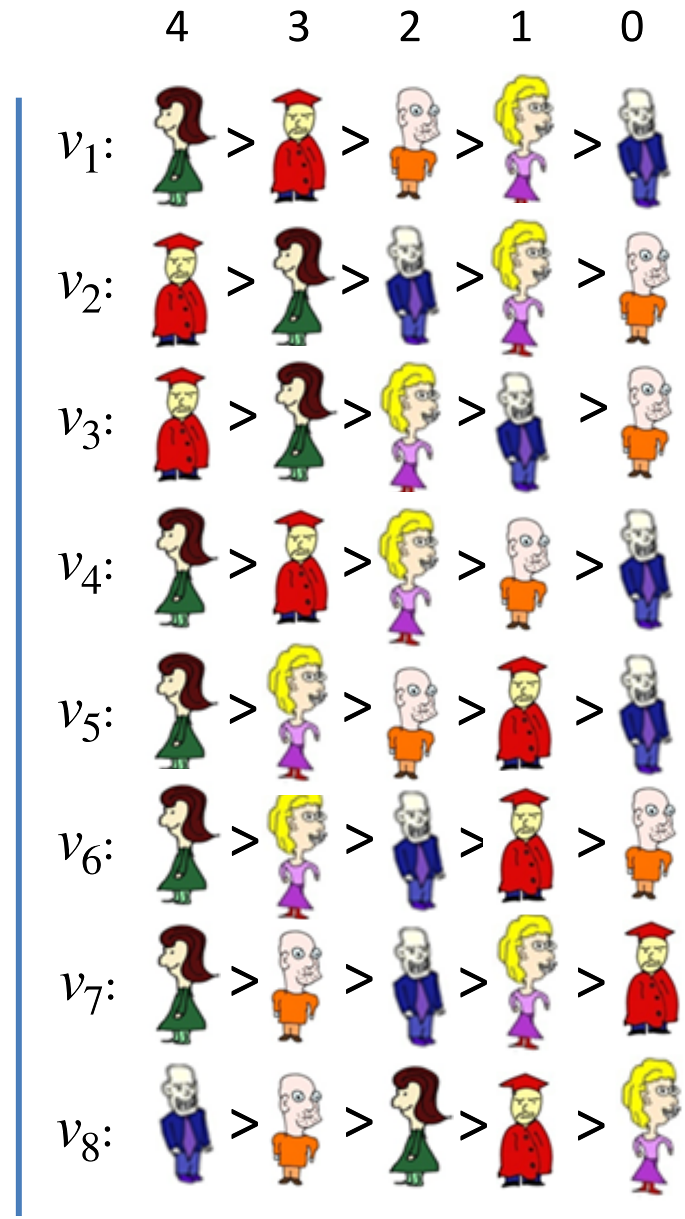
A committee with the best optimal valid assignment is winning.



The Greedy Monroe rule

Repeat k times:

1. Find a group G of n/k voters and a candidate c such that the score of voters from G from c is maximal.
2. Remove candidate c and the voters from G from the election.



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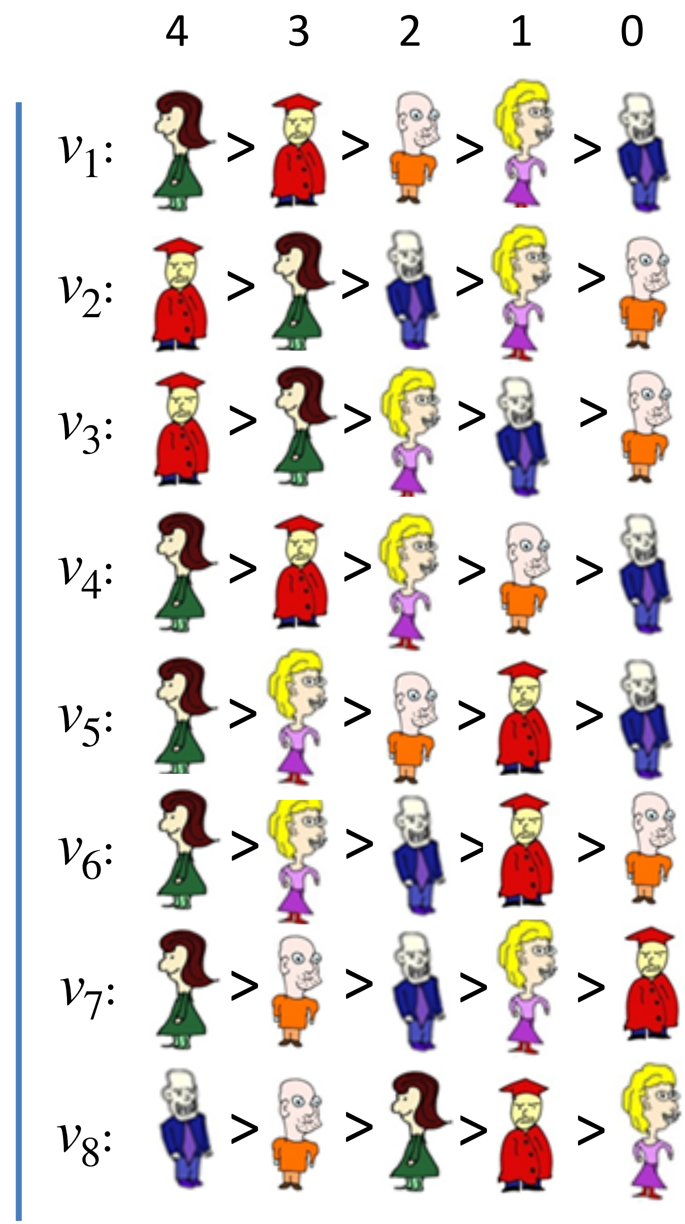

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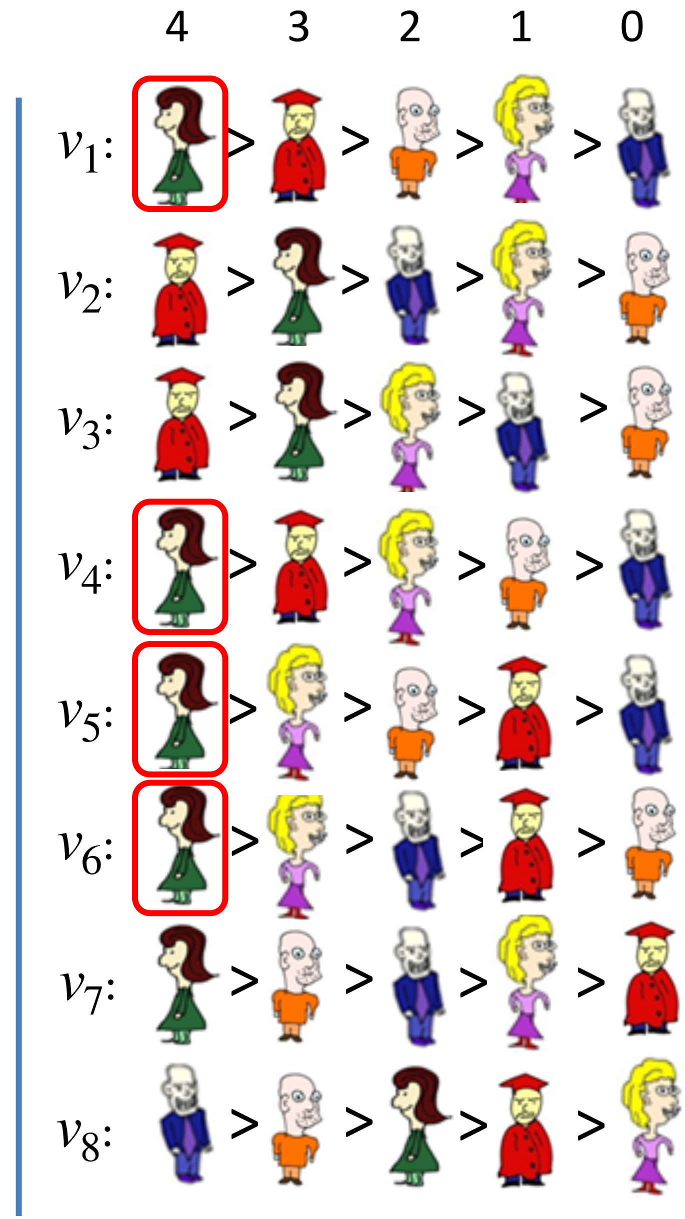

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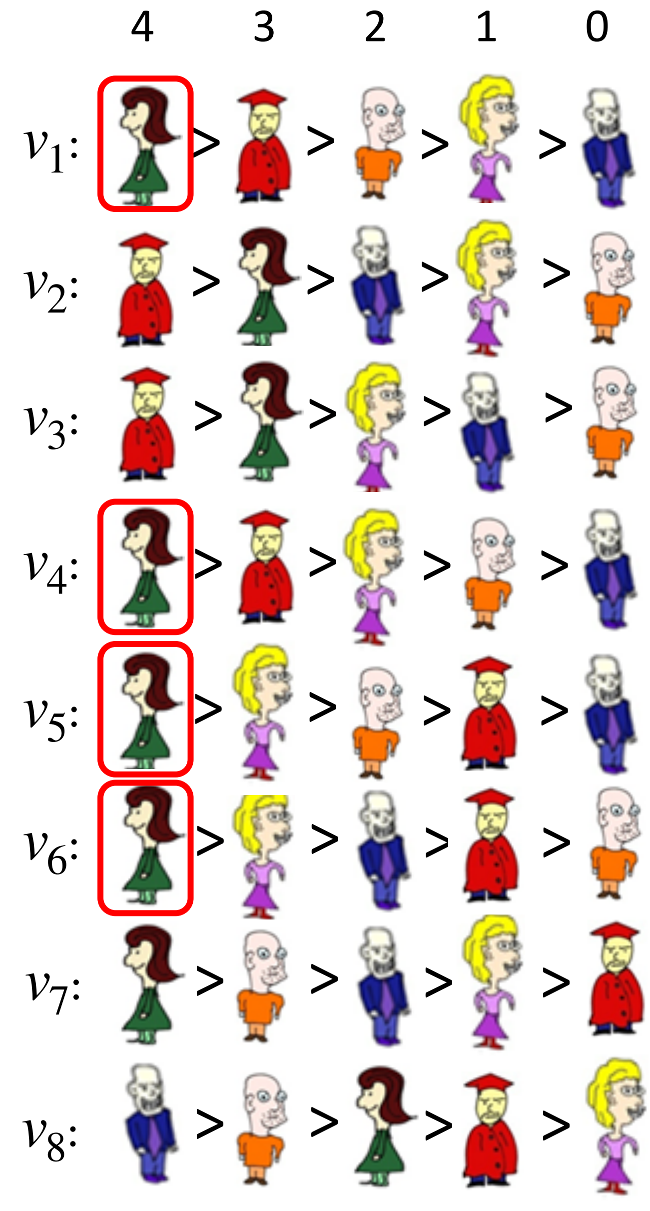

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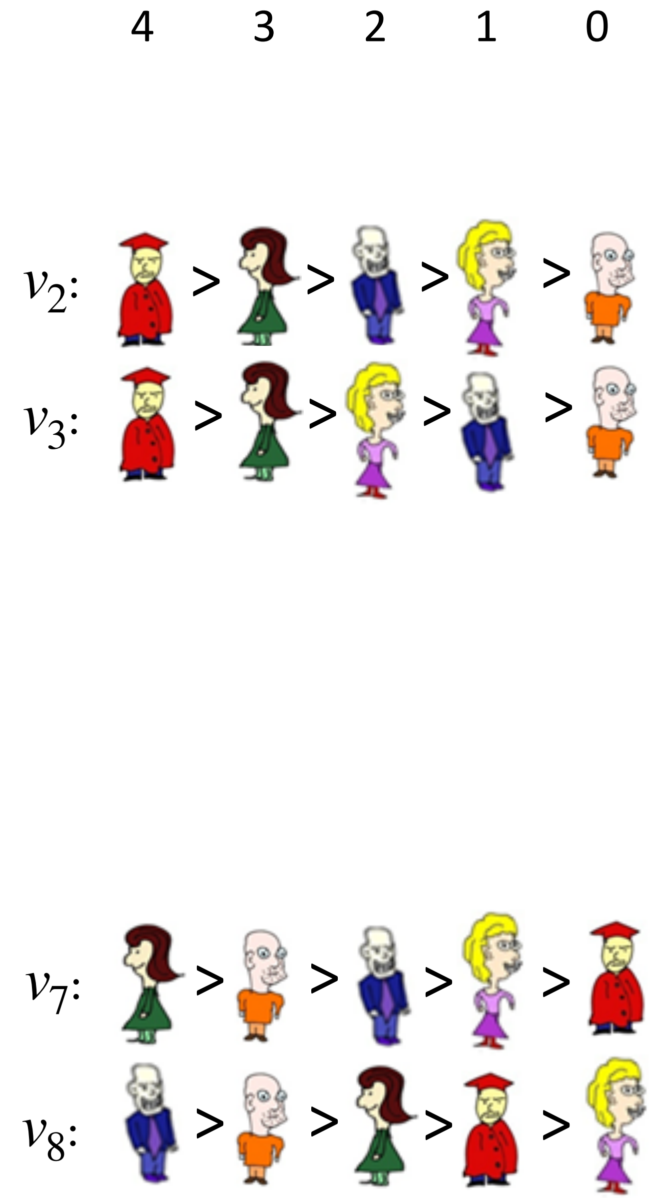

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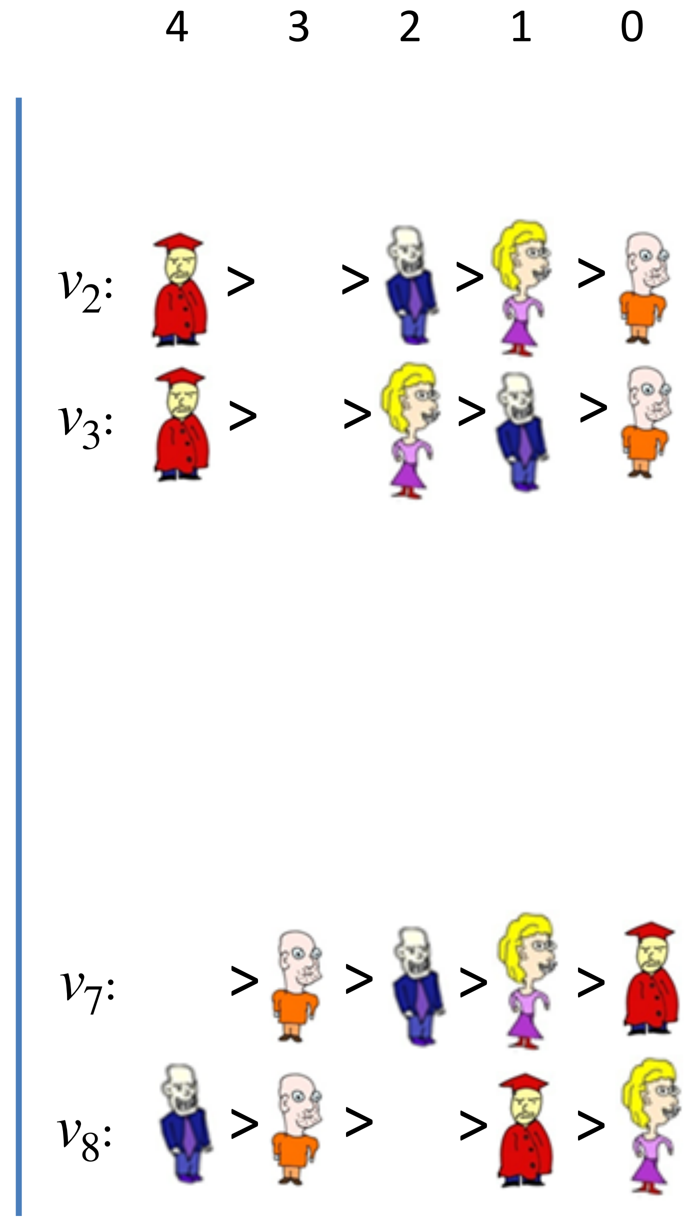

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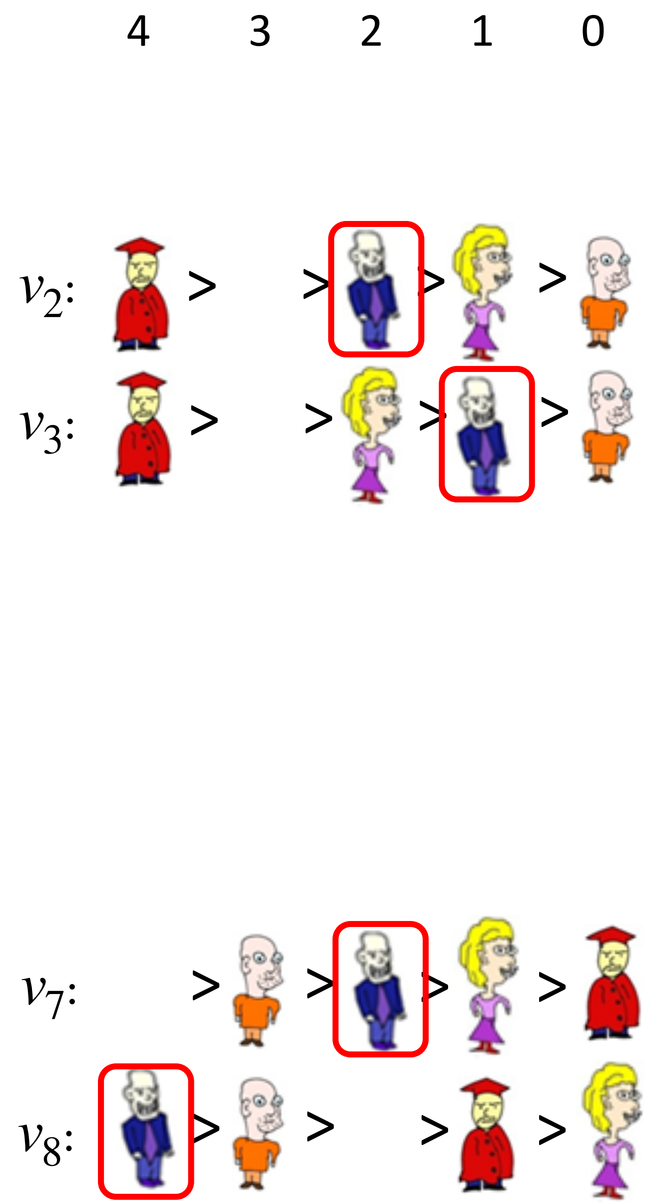

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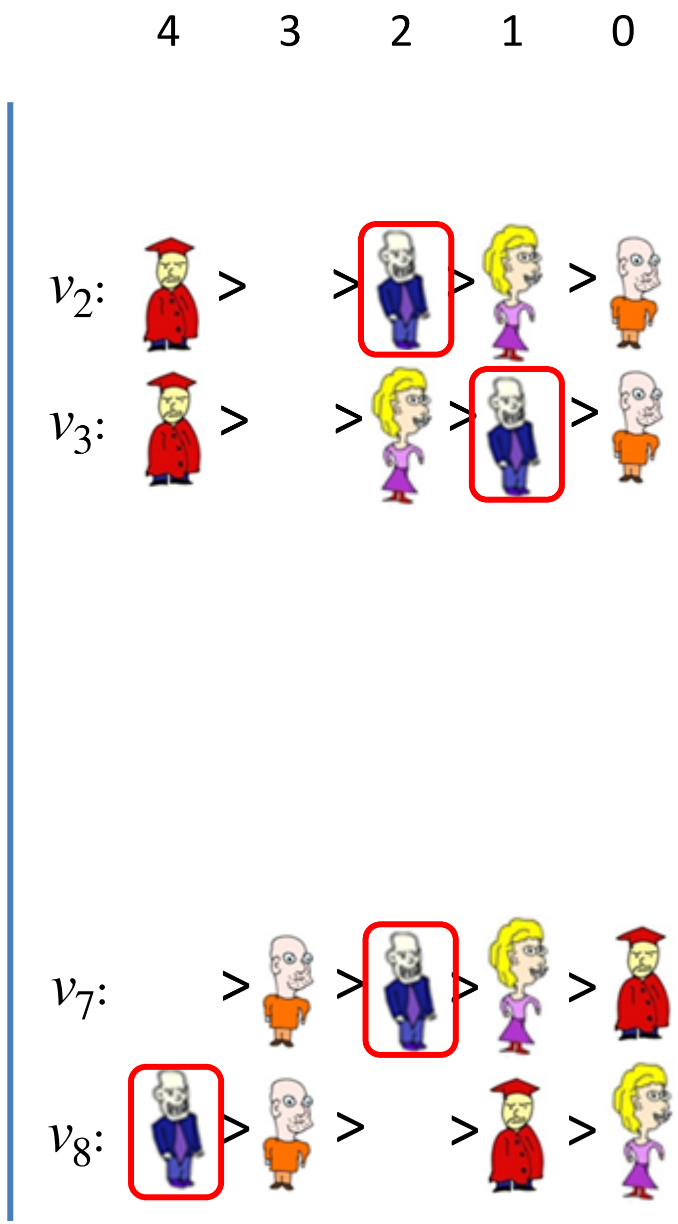
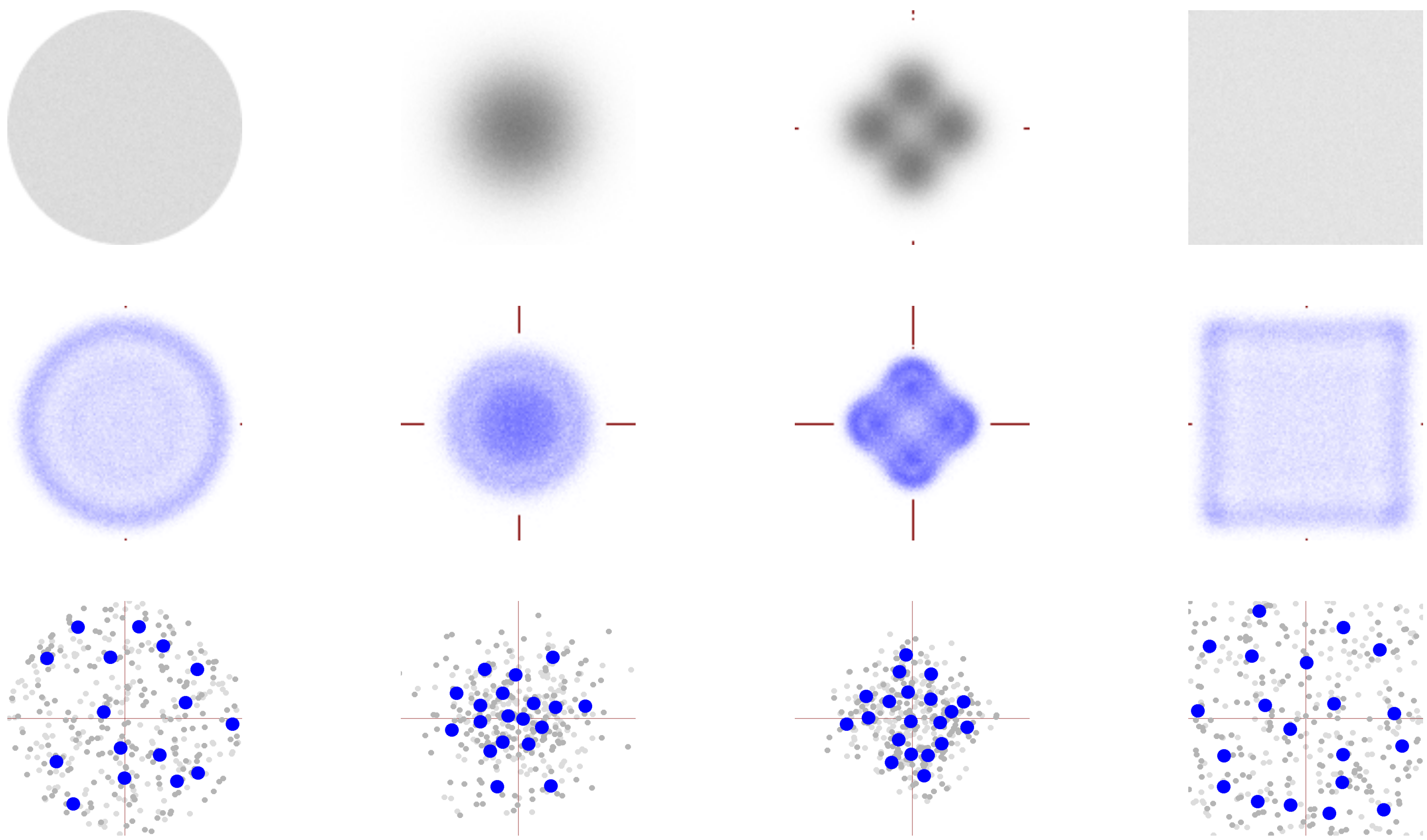


Monroe for 2-Euclidean Preferences

uniform on a circle | Gaussian | 4 Gaussians | uniform on a square



Greedy Monroe for 2-Euclidean Preferences

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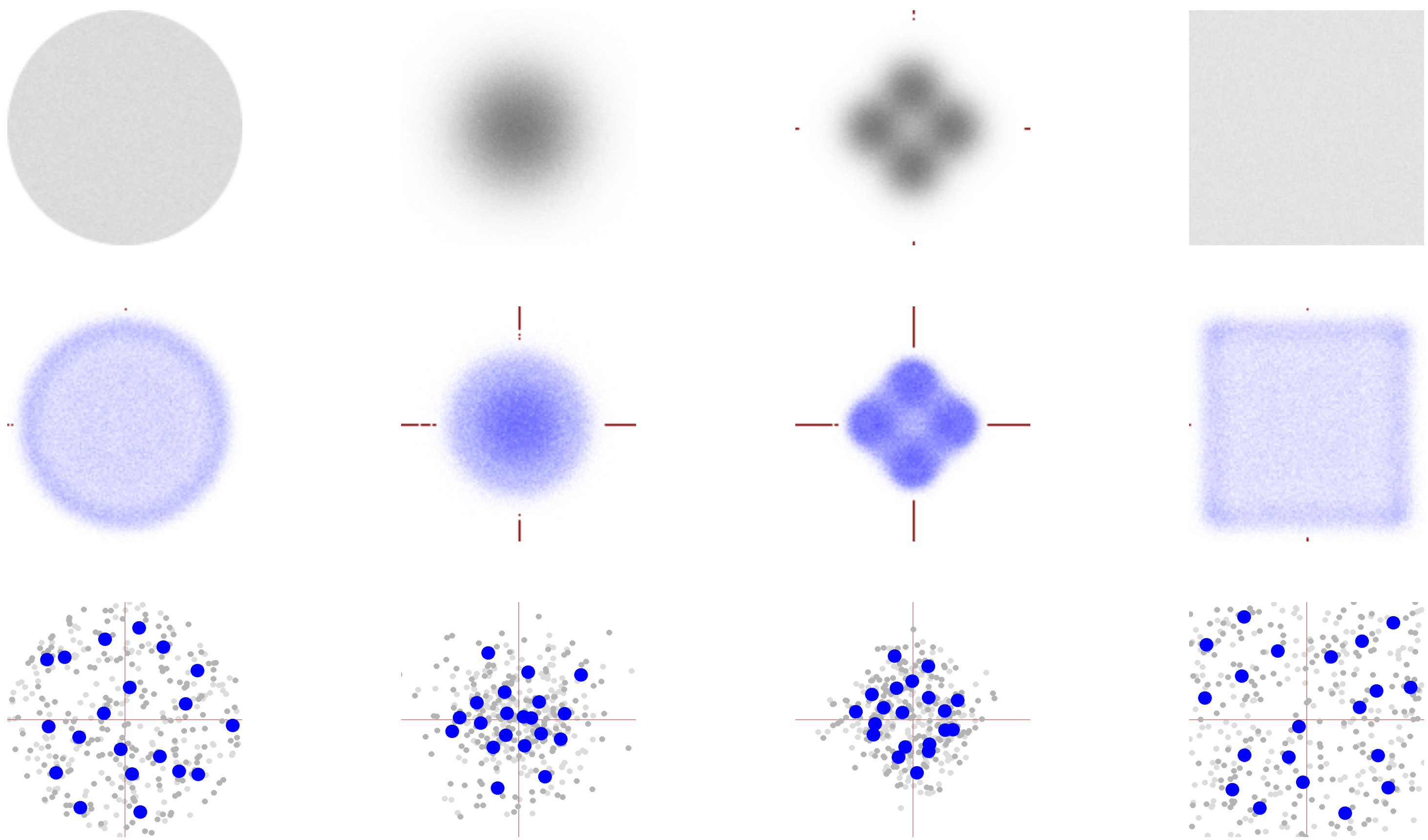
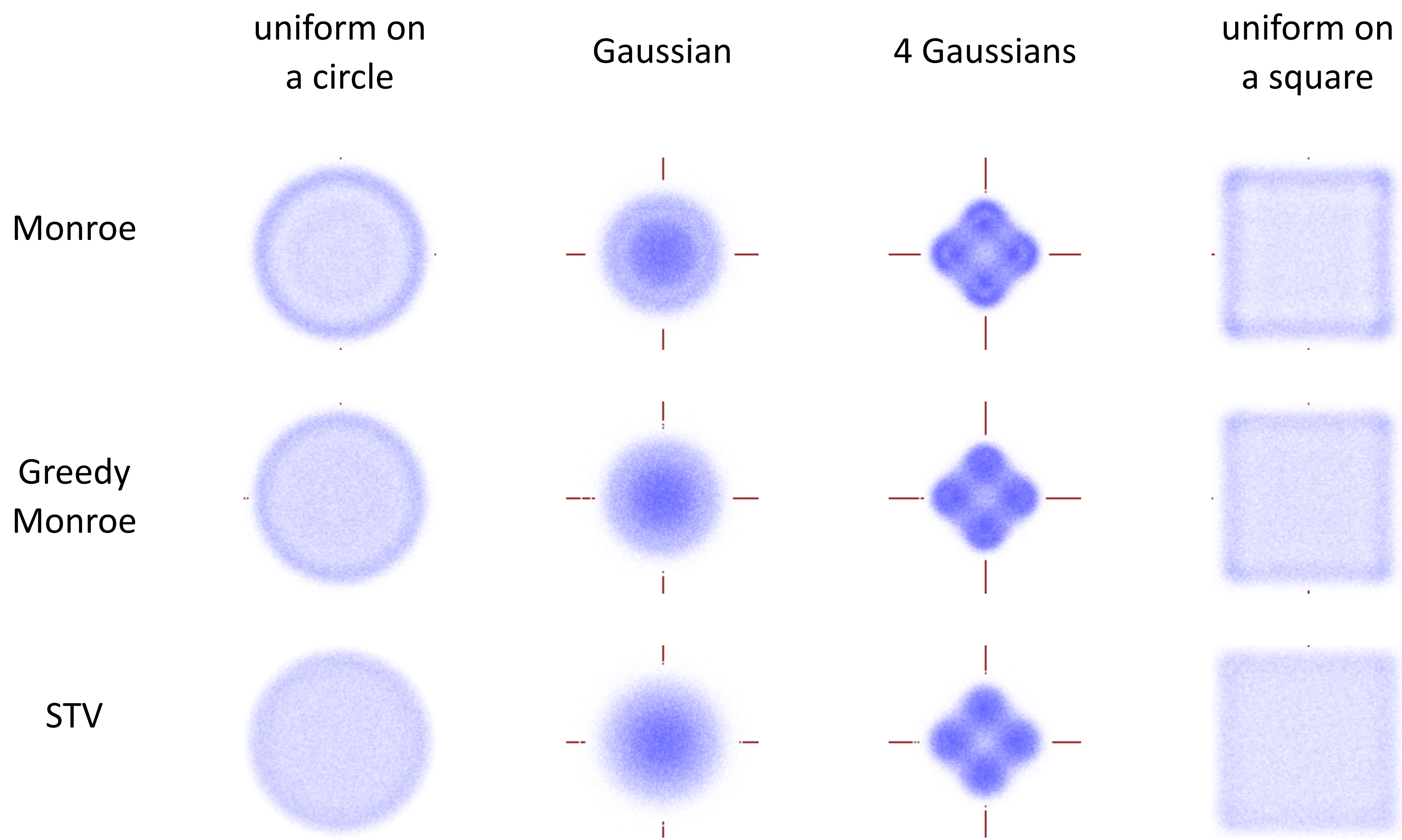


2-Euclidean Preferences: comparison



How Good is the Greedy Monroe Rule?

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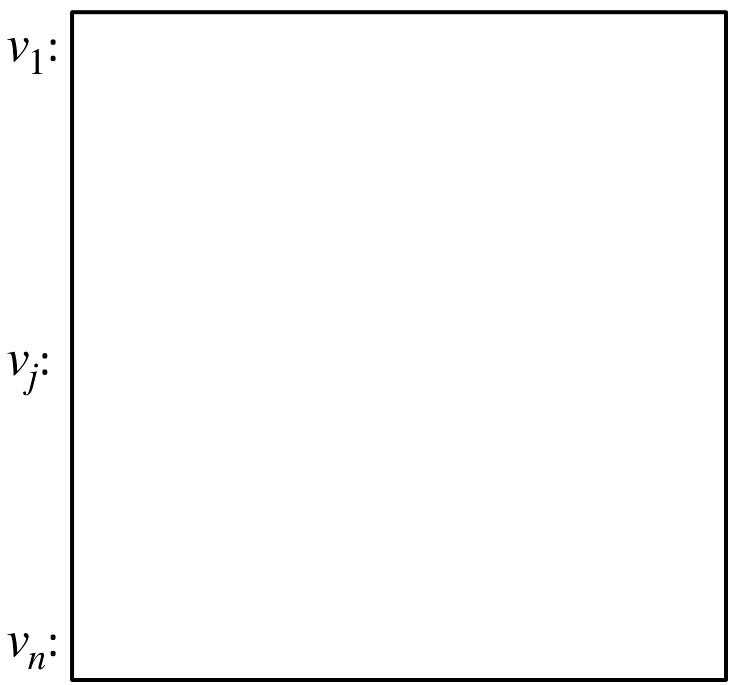
Consider the situation right after the i-th iteration



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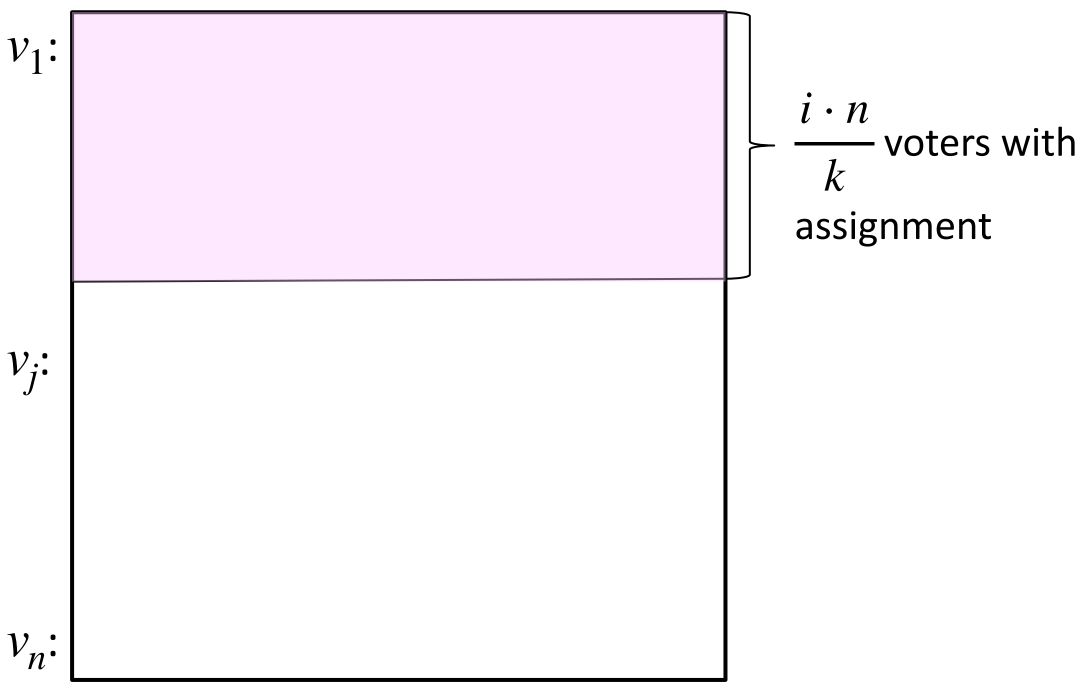

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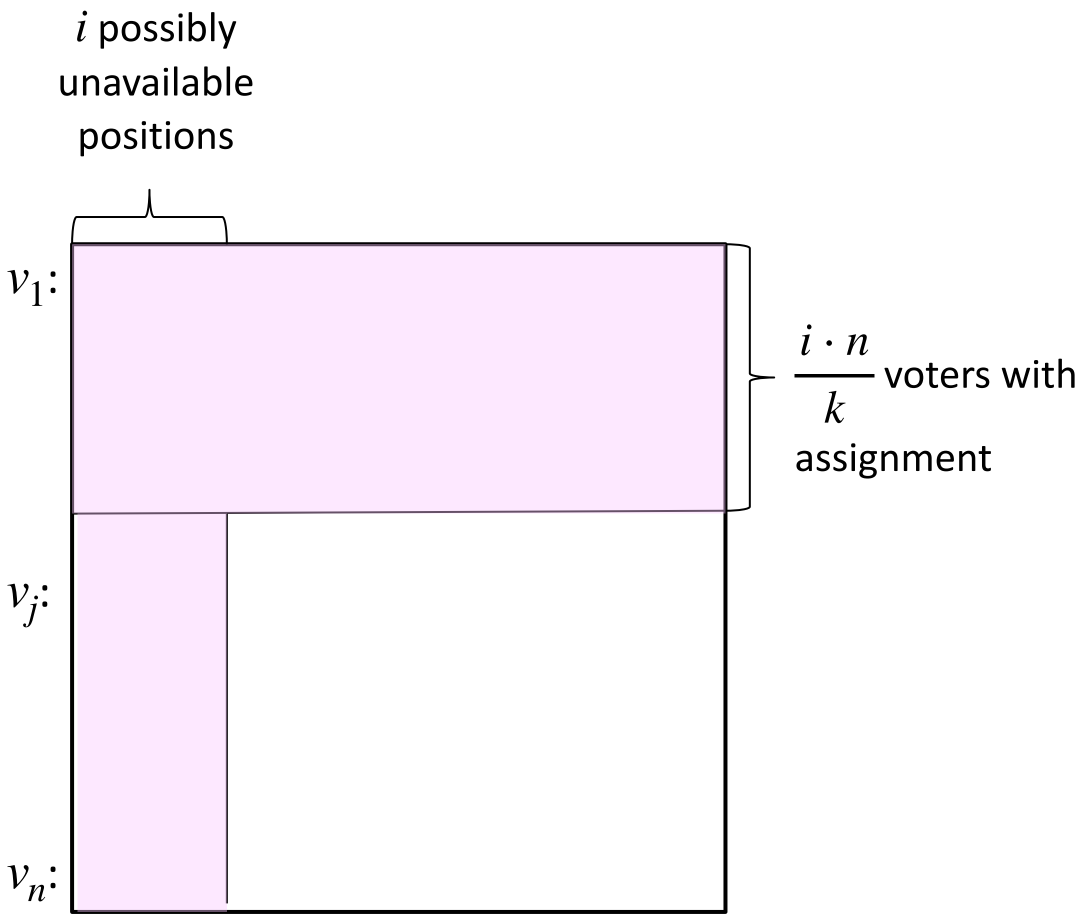

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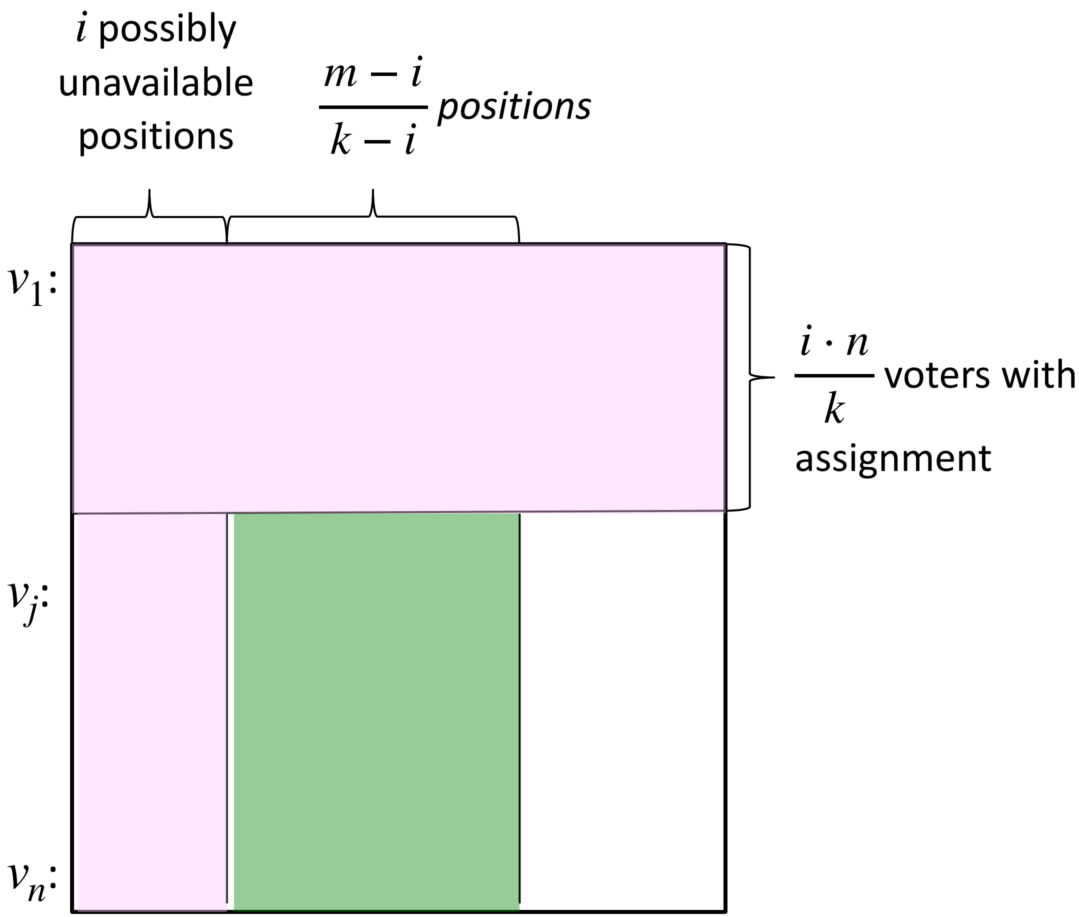

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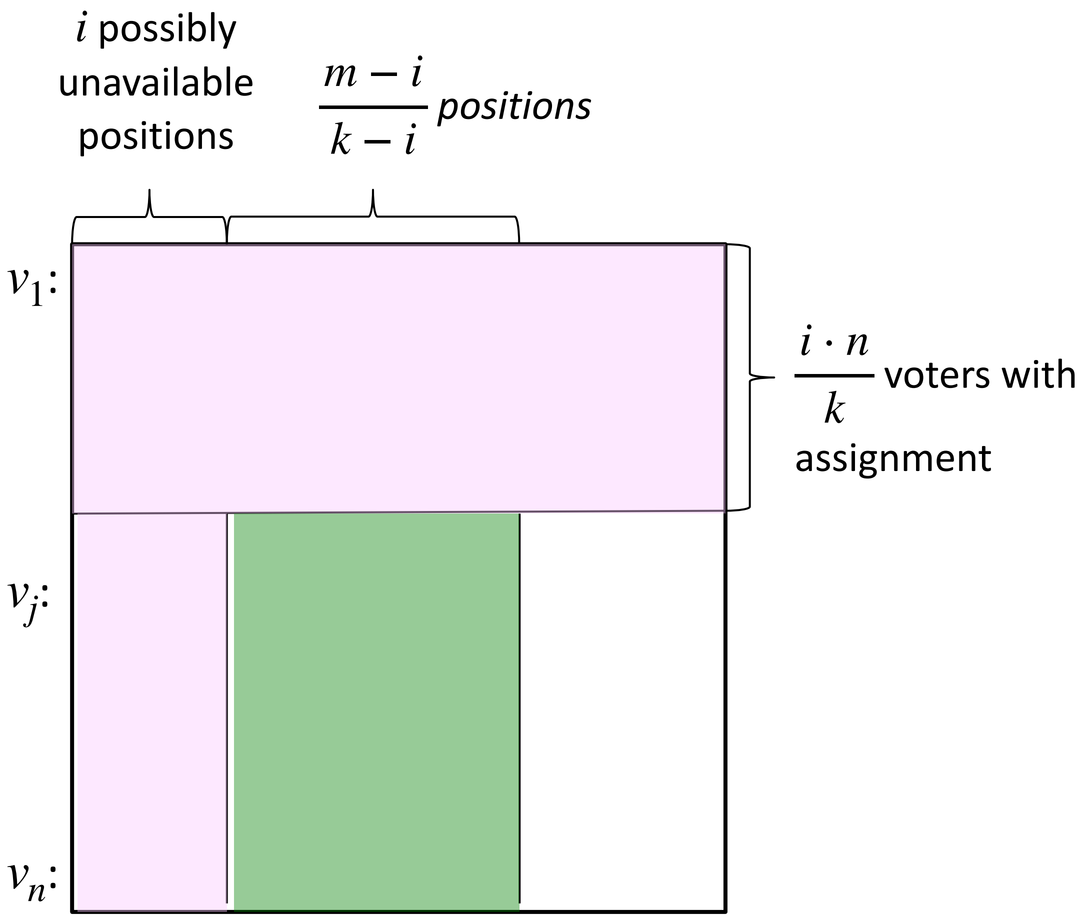

By pigeonhole principle, there is an unassigned candidate that n/k voters rank within the green area

$$\frac{1}{m-i}\left(\left(n-\frac{i \cdot n}{k}\right)\left\lceil\frac{m-i}{k-i}\right\rceil\right)$$

$$\geq \frac{n}{m-i} \cdot \frac{k-i}{k} \cdot \frac{m-i}{k-i} = \frac{n}{k}$$

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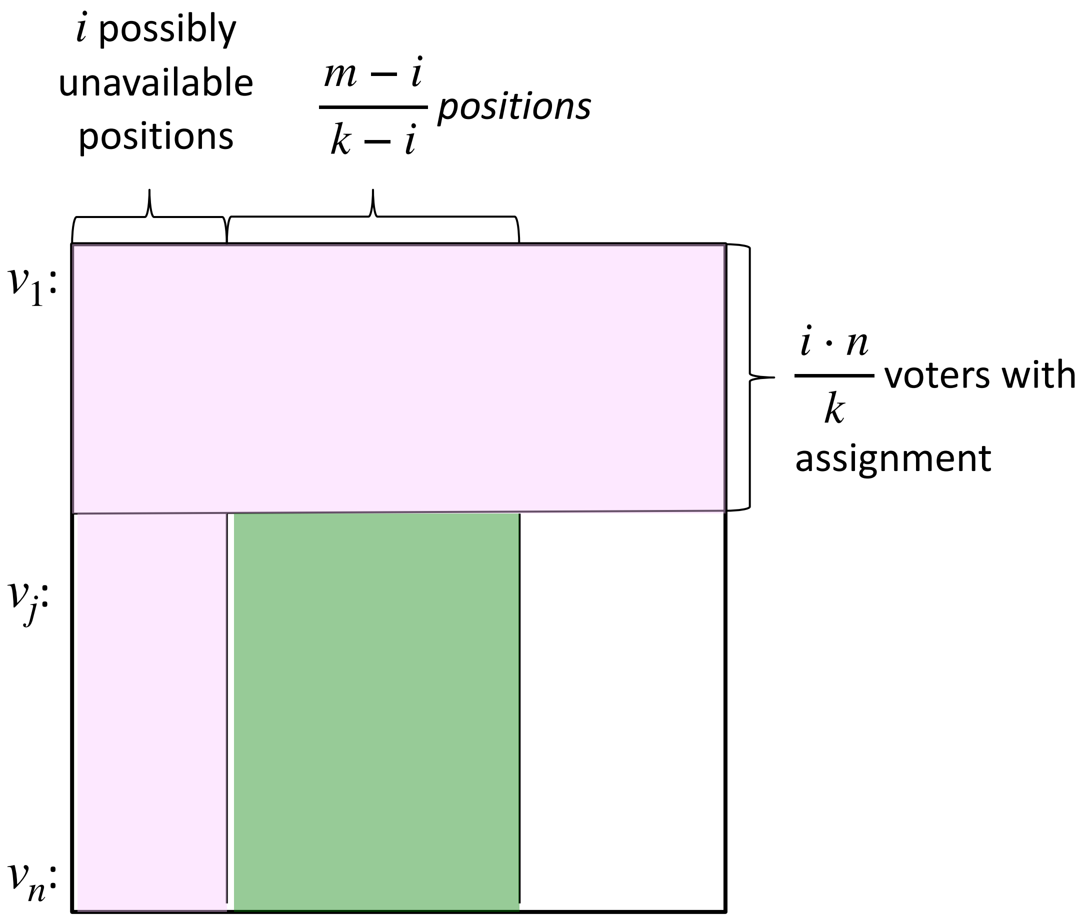

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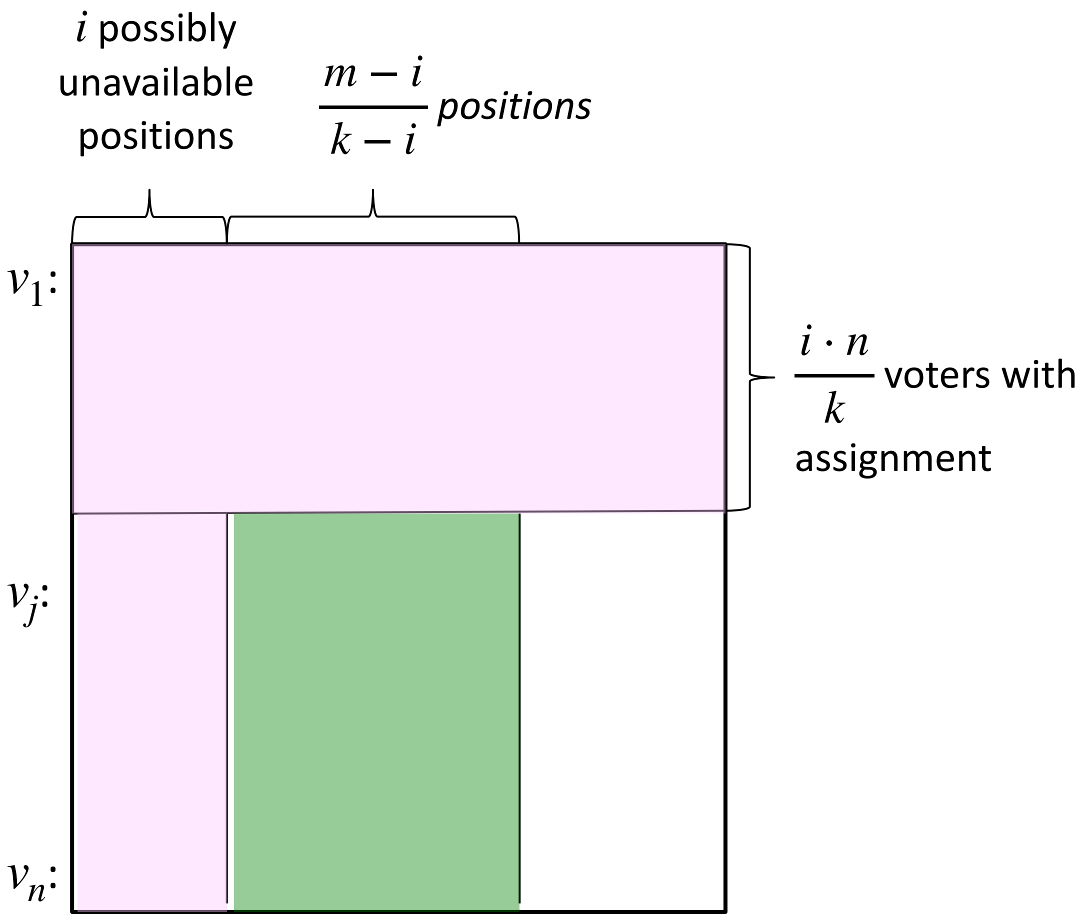

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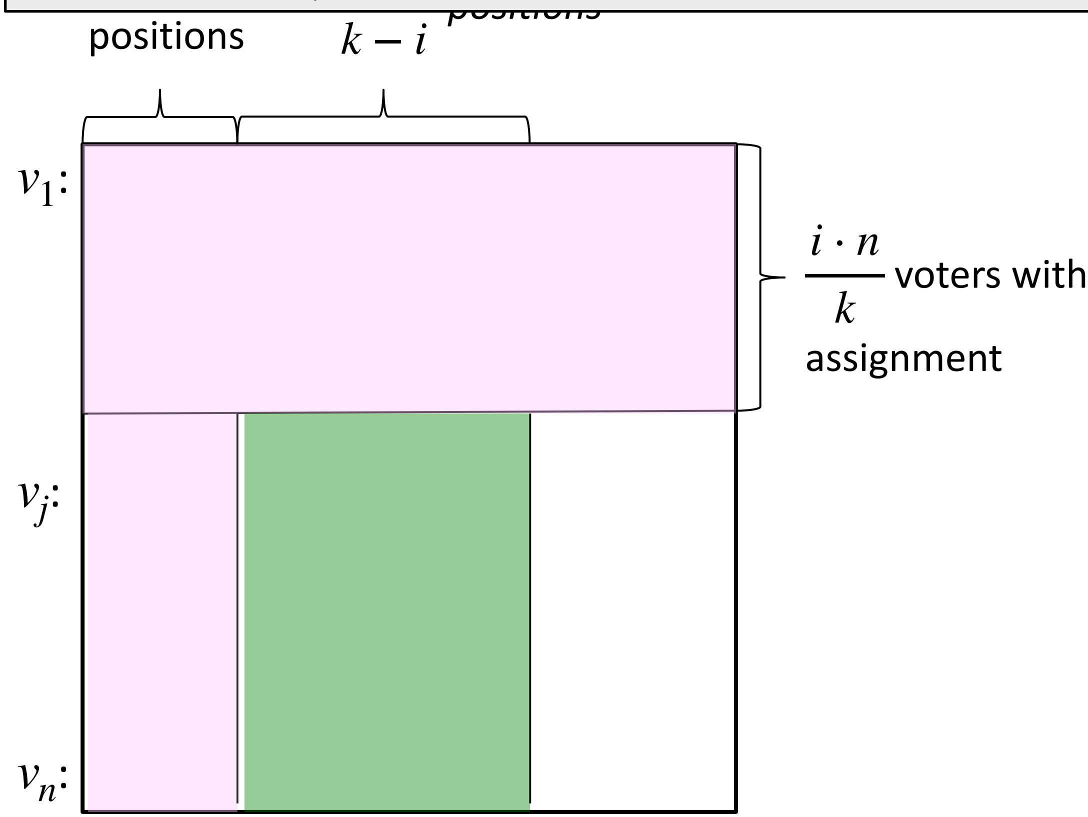

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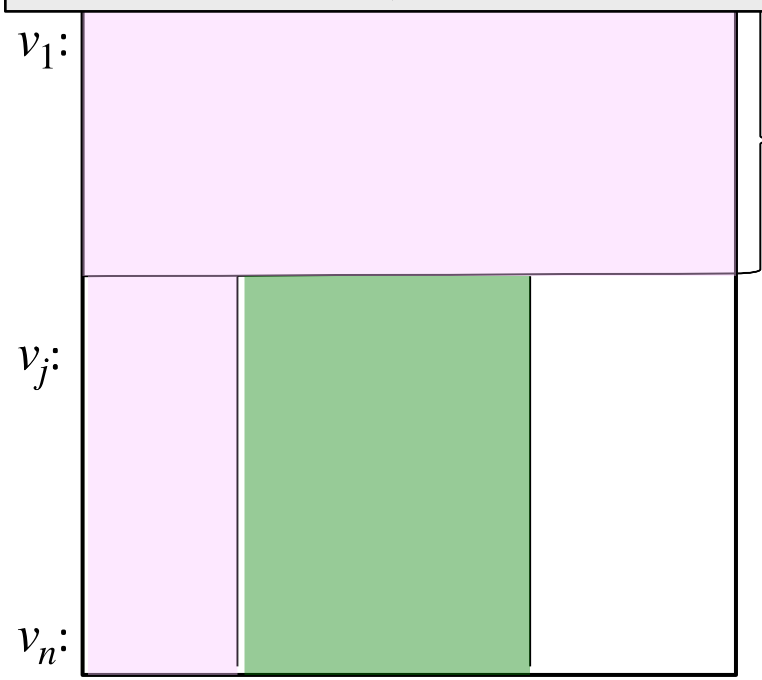

$\frac{i \cdot n}{k}$ voters with assignment

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$$= (m-1)n\left(1 - \frac{k-1}{2(m-1)} - \frac{\mathrm{H}(k)}{k} + \frac{\mathrm{H}(k)-1}{m-1} - \frac{\mathrm{H}(k)}{k(m-1)}\right)$$

C

e,

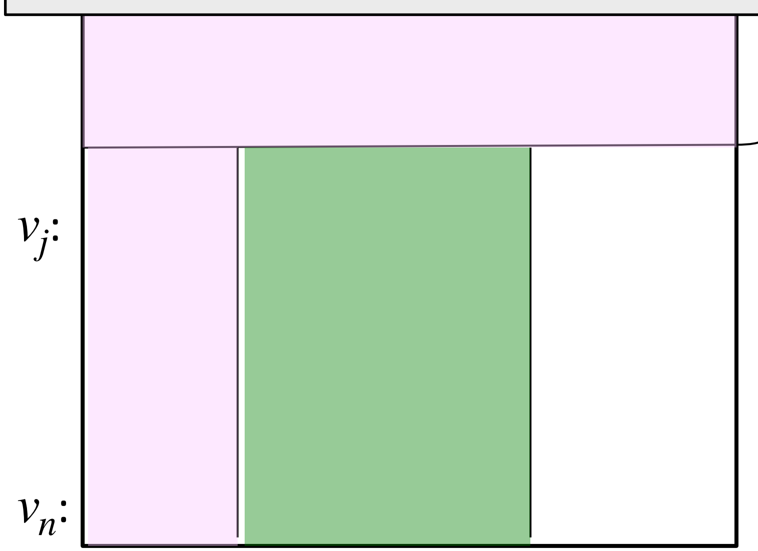
ers

rank within the green area

voters with

k

assignment



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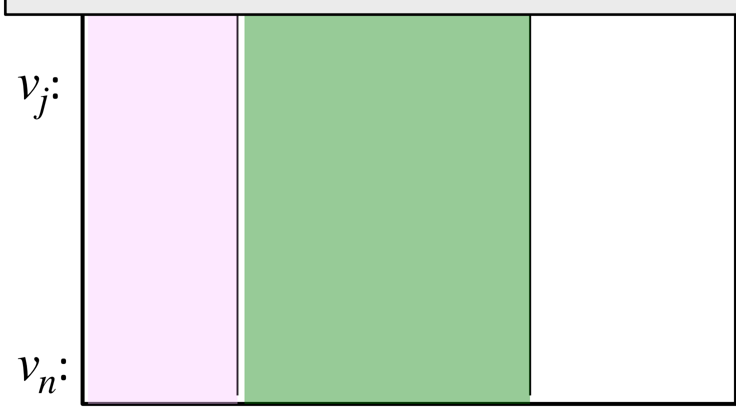

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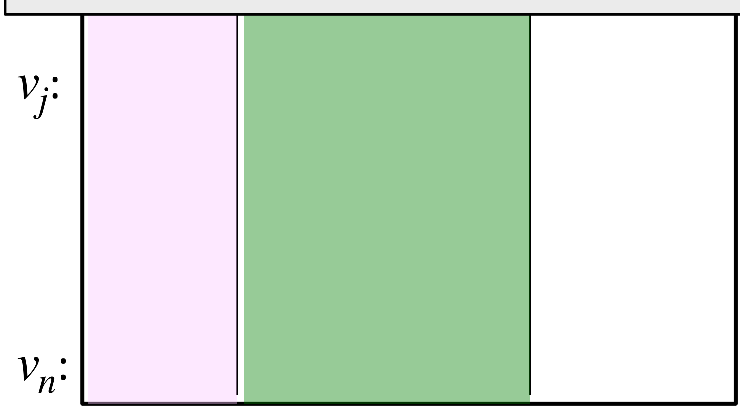

We achieve: **1 – (k-1)/2(m-1) – H_k/k** fraction of maximum possible satisfaction!

Okay, but is it really a good result?