PhD Open
Course “Proportional Algorithms”
Assignment

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You can choose 3 out of the following 4 problems.

Problem 1. Consider the Sainte-Laguë method of apportionment. The method works similarly to the D'Hondt method, but in the i-th round, it assigns a seat to the party j that has the maximum value of:

$$\frac{n_j}{2s_j(i)+1},$$

where:

- n_j is the number of votes received by party j, and
- $s_j(i)$ is the number of seats already assigned to party j up to round i.

Questions:

1. Show that the Sainte-Laguë method fails to satisfy the lower quota axiom.
2. Analyze how well this method approximates the lower quota. Specifically, determine the largest value of α such that a party with n_j votes is guaranteed to receive at least:

   $$\left\lfloor \frac{n_j}{n} \cdot k\alpha \right\rfloor$$

   seats, where n is the total number of votes and k is the total number of seats.

Problem 2. Consider the following rule, which is an adaptation of PAV (Proportional Approval Voting). Let A_i denote the approval set of voter i. The rule SLAV selects a subset W of size k that maximizes the following objective:

$$\sum_{i \in N} \sum_{j=1}^{|A_i \cap W|} \frac{1}{2j-1},$$

where N is the set of voters.

We say that a committee W satisfies α-EJR (Extended Justified Representation with factor α) if, for each ℓ-cohesive group of voters, there exists at least one voter i such that:

$$|A_i \cap W| \geq \lfloor \alpha \ell \rfloor.$$

What is the largest value of α such that the rule SLAV satisfies α-EJR?

Problem 3. Consider the following rule for ordinal preferences. In round i, $1 \leq i \leq m$ (m is the number of candidates), we proceed as follows:

1. If there exists a not-yet-selected candidate who is ranked among the top i positions by at least $\frac{n}{k}$ voters, we select this candidate.
2. After selecting such a candidate, we remove $\frac{n}{k}$ voters (chosen arbitrarily) who rank this candidate in their top i positions.
3. If such candidate does not exist, we move to the next round, $i := i + 1$.

Prove or disprove the following statements by providing either a proof or a counterexample:

1. The above rule satisfies proportionality for solid coalitions.
2. The rule satisfies candidate monotonicity (pushing a candidate higher in some voters' rankings cannot harm this candidate).

Problem 4. Assume that voters' preferences are given as rankings.

1. Is it possible to define utilities based on these rankings so that the Method of Equal Shares satisfies the axiom of proportionality for solid coalitions?
2. How can we formulate this axiom in the context of participatory budgeting, where candidates (projects) may have different prices?