Word Equations and Straight-Line Programs
Lecture 1: Satisfiability of Word Equations

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- Σ (finite) set of letters
- $\mathcal{X}$ (finite) set of variables

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  extend to $S : (\mathcal{X} \cup \Sigma)^* \to \Sigma^*$
- solution: $S(U) = S(V)$ (solution word)
- length-minimal solution: the one minimizing $|S(U)|$
  (other notions are OK as well)

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$aXXX = XaYYY$ has a solution $S(X) = a^{3n}, S(Y) = a^{2n}$

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Extensions

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- constraints (on $S(X)$, on $S(U)$)
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- satisfiability
- independence of the system
- nontriviality of solutions
- relations definability

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Satisfiability

- Markov: Hilbert 10th problem $\geq_r$ word equations (exercise)
- wanted to show undecidability

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Wrong

Decidable [Makanin '77]. Very complex.

Compression and word equations

Plandowski & Rytter '98

A length-minimal solution of size N a word equation of size n has a compressed representation of size $poly(n, \log N)$.

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A length-minimal solution of size N a word equation of size n has a Straight Line Program of size $poly(n, \log N)$.

Straight Line Program (SLP)

A context free grammar generating exactly one word.

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Guess compressed representation of variables and verify the guess.
Depends on external guarantees for N (Makanin's algorithm).

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Remark

Compress and compute paradigm turns out to be useful all around.

SLP, Composition system

SLP

Context free grammar generating a single word
Usually: several natural restrictions

- rules $A \to BC$ or $A \to a$ (Chomsky normal form)
- no chain rules ($A \to B$)
- no ϵ-rules ($A \to \epsilon$)
- only useful rules (used in the derivation)
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Generalized SLP: Composition systems

Productions of the form

$$X \to Y[i\,..\,j]Z[i'\,..\,j']$$

A natural meaning: $Y[i\,..\,j]$ is a substring of word generated by Y.
Can be transformed to SLP with a quadratic blow-up.

Plandowski & Rytter: Idea

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There are $\leq |U| + |V|$ cuts.

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Main observation

If S is length-minimal and w occurs in $S(u)$, then it "crosses" a cut.
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Main observation

If S is length-minimal and w occurs in $S(u)$, then it "crosses" a cut.
(i.e. the cut is inside or at one of the ends)
Otherwise all occurrences could be removed.

Plandowski & Rytter proof

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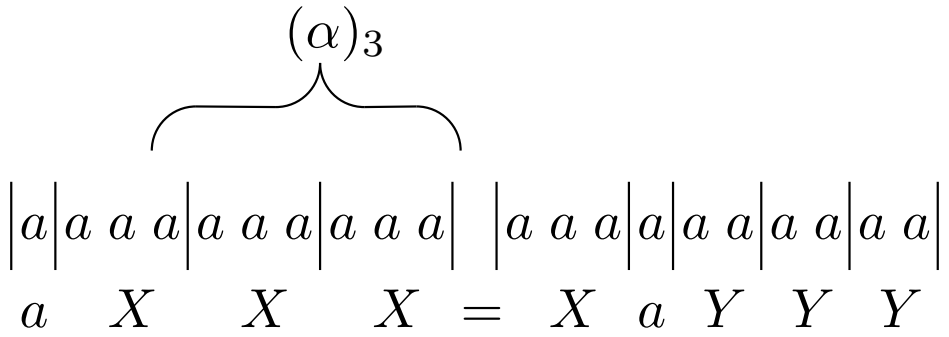

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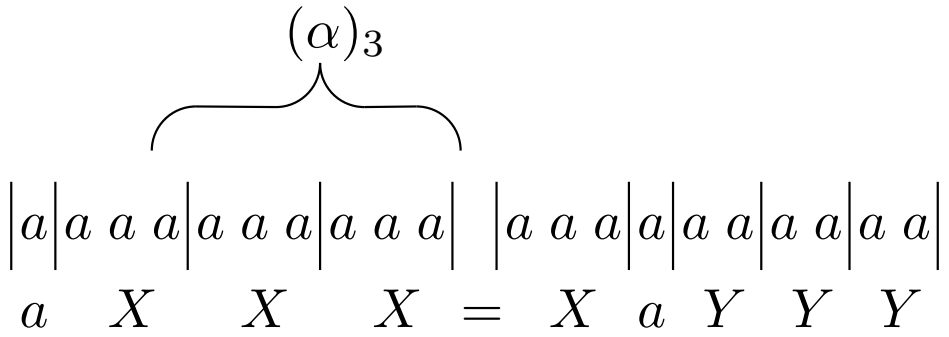

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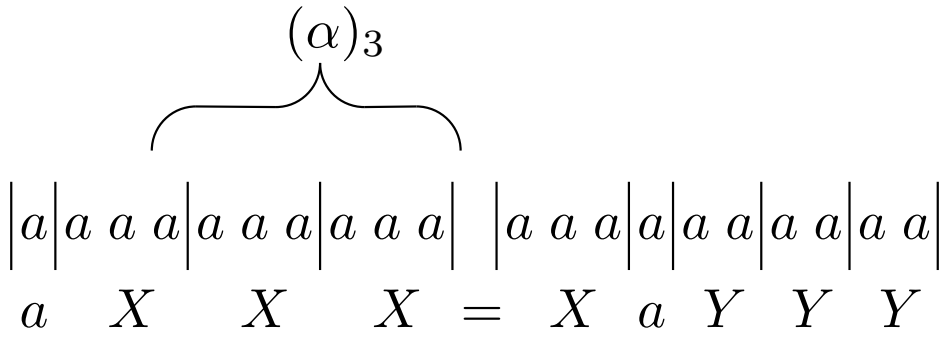

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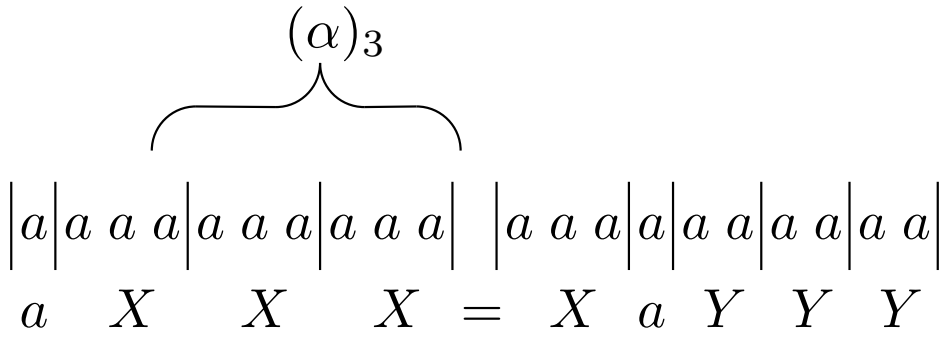

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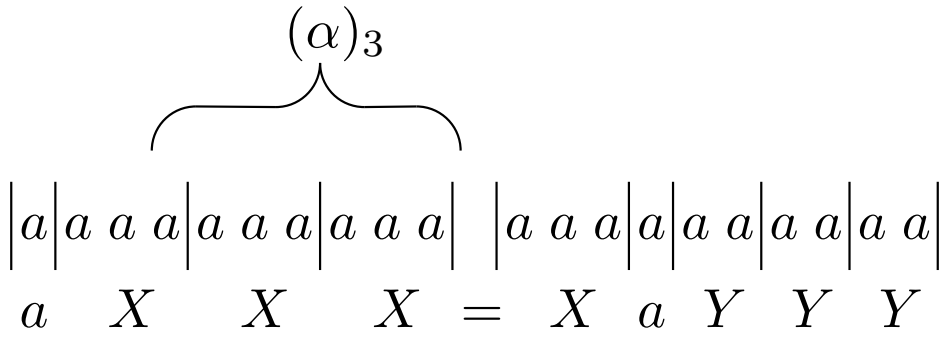

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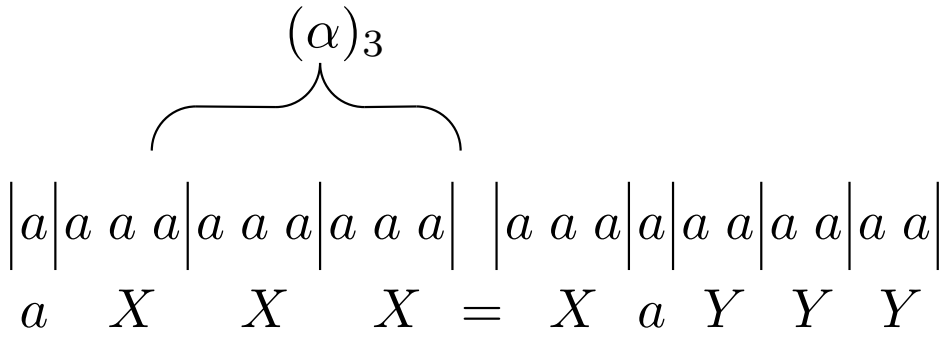

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SLP: size $poly((|U| + |V|)\log N)$.

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Theorem (Plandowski 1999)

Length-minimal solution is at most doubly exponential.

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We prove the existence of the SLP by constructing it top-down.

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We prove the existence of the SLP by constructing it top-down.

Why not bottom-up?

There has to be a rule $X \to ab$.
We can build SLP bottom-up.

Equality and Compression of Words

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Intuition: recompression

- New letters: nonterminals of a grammar
- We build a grammar for both words, bottom-up.
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Comparison with Plandowski's approach

Plandowski & Rytter: Top-down construction; different SLPs for sides
Plandowski: exploits structural properties, many technical problems.

Algorithm: idea

Choose a pair (or letter) from $S(U)$ and compress it.

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while $U \notin \Sigma$ and $V \notin \Sigma$ **do**
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How to do this for equations?

Idea at work

Working example

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$$\begin{aligned} XbaYb &= baaababbbab \quad \text{for } S(X) = baaa\ S(Y) = bba \\ XcYb &= caacbcb \quad \text{for } S'(X) = caa\ S'(Y) = bc \end{aligned}$$

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There is a problem with 'crossing pairs'. We will fix!

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So we can assume that:

- there are (≥ 2 different) letters in the equation
- the solution is over the letters in the equation

Pair types

Definition (Pair types)

Occurrence of ab in $S(U)$, $S(V)$ (for a fixed substitution S) is

- explicit: it comes from U or V;
- implicit: comes solely from $S(X)$;
- crossing: in other case.

ab is crossing if it has a crossing occurrence, non-crossing otherwise.

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Compression of non-crossing pairs

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Lemma

The PairComp(a, b) *properly compresses noncrossing pairs.*

Correctness of nondeterministic procedures

Definition (Soundness, completeness)

A nondeterministic procedure is
sound: for all nondeterministic choices: if the returned equation is satisfiable $\implies$ the original one is satisfiable
complete: given a satisfiable equation for some nondeterministic choices it returns a satisfiable equation.

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In this case

If $U = V$ has a solution S such that ab is non-crossing then the returned equation $U' = V'$ has a solution S' such that $S'(U')$ is $S(U)$ with each ab replaced with c.

Correctness

Complete

$S'(X) \leftarrow S(X)$ with each ab replaced with c
$S'(U')$ is $S(U)$ with every ab replaced; similarly $S'(V')$:

explicit pairs replaced explicitly

implicit pairs replaced implicitly (in the solution)

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$$X\ c\ a\ Y\ b = c\ aa\ c\ ab\ c\ b \qquad S'(X) = caa\ S'(Y) = bc$$

Correctness

Complete

$S'(X) \leftarrow S(X)$ with each ab replaced with c
$S'(U')$ is $S(U)$ with every ab replaced; similarly $S'(V')$:

explicit pairs replaced explicitly

implicit pairs replaced implicitly (in the solution)

crossing there are none

Sound

If the new equation is satisfiable: roll back the changes.

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Dealing with crossing pairs

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There is X such that $S(X) = bw$ and aX occurs in $U = V$ (or symmetric).

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Uncrossing(a, b)

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2: **if** first letter of $S(X)$ is b **then**
3: replace each occurrence of X by bX ▷ Pop
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Lemma

Uncrossing is sound and complete.
If S is a solution of $U = V$ then for appropriate nondeterministic choices the returned equation $U' = V'$ has a solution S' such that $S(U) = S'(U')$ and ab is non-crossing.

Uncrossing: example

Uncrossing *ab*

$$X \;\; baa \;\; Y \;\; b{=}baaabaabbbab \quad S(X) = baaa \; S(Y) = bba$$

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Proof

Let $a_X \in \{\varepsilon, a\}, b_X \in \{b, \varepsilon\}$, such that we replaced X with $b_X X a_X$ (or $b_X a_X$).
Sound: if S' is a solution, define $S(X) = b_X S'(X) a_X$.
Then $S(U) = S'(U')$.

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Complete: We make the choices according to S:

- if a is last letter of $S(X)$ then replace X with Xa ($a_X = a \iff S(X)$ ends with a)
- if b is first letter of $S(X)$ then replace X with bX ($b_X = b \iff S(X)$ begins with b)
- if $S(X) = \varepsilon$ then remove it

Define $S'(X)$ such that $S(X) = b_X S'(X) a_X$.

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Hence $S(U) = S'(U')$.

Can ab be crossing?

If $b_X = \varepsilon$ then first letter of $S'(X)$ is not b: cannot create crossing pair.

If $b_X = b$ then the letter to the left of X is $b \neq a$: cannot create a crossing pair.

Maximal blocks

Definition (maximal block of a)

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Lemma (Exponent of periodicity: simple case)

If a^ℓ is a maximal block in a length-minimal solution of $U = V$ then $\ell \leq 2^{c|UV|}$.

Blocks compression

When a has no crossing block

1: **for** all maximal blocks a^ℓ of a and $\ell > 1$ **do**
2: let $a_\ell \in \Sigma$ be an unused letter
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Lemma

The ChainNCr(a) *is sound and complete.*
If $U = V$ has a solution S such that a is non-crossing then the returned equation $U' = V'$ has a solution S' such that $S'(U')$ is $S(U)$ with each maximal a-block a^ℓ is replaced with a_ℓ.

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Lemma

a-block uncrossing is sound and complete.
If S is a solution of $U = V$ then for appropriate nondeterministic choices the returned equation $U' = V'$ has a solution S' such that $S(U) = S'(U')$ and a is non-crossing.

Algorithm

while $U \notin \Sigma$ and $V \notin \Sigma$ **do**
    $L \leftarrow$ letters from $U = V$
    choose a pair of letters or a block from L
    if it is crossing **then**
        Uncross it
    Compress it

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- Composition of sound (complete) procedures are sound (complete) for every choice of letter/pair!
- we have a choice of pair/letter

Equation's size

We show that

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- After each operation the length-minimal solution shortens.

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So we terminate on positive instances.
(we do not accept negative by soundness)

Strategy

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Each compression decreases the length of the length-minimal solution word.

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We perform the compression on the solution word. □

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Something is compressed in the equation. □

Strategy

- If there is something non-crossing: compress it.
- If there are only crossing: choose one that minimises the equation.

Comments

- for a given equation we can “choose” an arbitrary solution and perform compression according to it
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- the new equation has a corresponding (compressed) solution; it may have other, shorter and longer
- if we "choose" a length-minimal one and then we can guarantee some compression rate
- for any choice of solution we guarantee the equation size (for appropriate choice of pair/letter)

Lemma (Fixed solution)

There are at most $2n$ different crossing pairs and blocks.

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Lemma

There is always some choice to be $\leq 8n^2$.

There are $m \leq 8n^2$ letters and $k \leq 2n$ different crossing blocks/pairs.
Some covers $\geq m/k$ letters.
Its compression removes $\geq m/2k$ letters and introduces $2n$ letters.
We are left with at most

$$(1 - 1/2k) \cdot m + 2n \leq (1 - 1/4n) \cdot 8n^2 + 2n = 8n^2 \ .$$

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- one can deduce doubly-exponential upper bound on solution size
- only exponential lower bound
- exponential upper bound $\implies$ NP-algorithm
- many believe in exponential upper bound

Extensions

Main advantage: robust.

- many things can be changed/altered
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Extensions

- regular constraints
- involution
- (free) groups
- representation of all solutions
- partial commutation (not so easy)
- term equations (not so easy)

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- for a regular langauge we define the (semigroup of) transition matrices (Boolean operations) of the NFA
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- constraint: we specify for a variable X the transition ρ_X and require that $\rho(S(X)) = \rho_X$

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Changes in the algorithm

- when popping, we need to guess new transition functions, i.e. when we pop a we need ρ'_X such that

$$\rho_X = \rho'_X \rho(a)$$

- everything more or less generalizes (bound on exponent of periodicity, correctness, etc.)

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Extend the alphabet: add a letter for each possible transition (oracle access!).
This does not change satisfiability.

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- We keep (as a tool of a proof) a mapping from current alphabet to strings over original alphabet.
- so the solution is over the current alphabet $\cup$ original alphabet.

Involution

Definition (Involution)

Think of it as the inverse in a free group.

Bijection $\bar{\cdot}$ on Σ^* and on $\mathcal{X}$ such that

- $\bar{\bar{a}} = a$
- $\overline{a_1 a_2 \cdots a_k} = \overline{a_k} \cdots \overline{a_2}\,\overline{a_1}$

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Used for

- groups (formal inverse)
- biology computation
- string operation (reversal)
- ...

How to deal with involution

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Remark

When we compress w to c we compress $\overline{w}$ to $\overline{c}$.
What if they overlap?

- some simple special cases ($ab\overline{a}$, etc.)
- $a = \overline{a}$, $b = \overline{b}$: we treat alternating a, b-strings as blocks

From groups to semigroups

Diekert, Gutiérrez & Hagenah 2001

Equations in free groups with regular constraints

Reduce to

Equations in free semigroups with regular constraints and involution

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Properties

- purely syntactical
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The same holds for other groups, but the obtained semigroups differ.

Reduction

- Triangulate the equation (equations $XY = Z$ and $X = a$)
- Remove cancellation: $XY = Z \quad \rightarrow$
  $X = X'R,\ Y = R^{-1}Y',\ X'Y' = Z$
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- Bijection on sets of solutions.
- We need involution and regular constraints

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Transforming the solutions

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Problem

No bound on the exponent in block compression.

$$aXXXX = XaYY \implies S(X) = a^{2m}, S(Y) = a^{3m}$$

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Transforming the solutions

The algorithm transforms the solutions:

- given an equation $U = V$ and a solution S, for some nondeterministic choices we obtain an equation $U' = V'$ with solution S' that "corresponds" to $S(U)$
  (it is the same or some compression applied)
- if an equation $U = V$ is transformed to $U' = V$ which has a solution S' then we can construct a corresponding solution S of $U = V$.
  (appending/prepending letters, $c \to ab$)

Problem

No bound on the exponent in block compression.

$$aXXXX = XaYY \implies S(X) = a^{2m}, S(Y) = a^{3m}$$

This boils down to solutions of systems of integer equations.
More about it later on.

Graph representation of all solutions

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- Graph (exponential size)
- Nodes: equations of size $\mathcal{O}(n^2)$, at most n occurrences of variables
- Edges between nodes, labelled with (families of) morphisms

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If S is a solution of $U = V$ then

- the equtaion is trivial or
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Each solution is obtained on a path from original equation to a trivial one.