Type-Based Synthesis and the Inhabitation Problem

Lecture 3: Inhabitation in Intersection Types and Bounded Combinatory Logic

Jakob Rehof
TU Dortmund University, Germany

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Intersection Types and Decision Problems

- *Intersection types* [CDC80; CDCV81; BCDC83] allow terms to be assigned multiple types $\Gamma \vdash M : \tau_1 \cap \ldots \cap \tau_n$
- Due to their enormous expressive power, the classical decision problems, *typability* and *inhabitation*, are undecidable
- Compare with simple types:
  - Typability is in linear time (unification)
  - Inhabitation (provability in minimal intuitionistic logic) is PSPACE-complete [Sta79]

Intersection Types

Definition ([CDCV80],[BCDC83], ..., [BDS13])

Types $\mathbb{T}$: $\sigma, \tau ::= a \mid \sigma \to \tau \mid \sigma \cap \tau$, Atoms $a \in \mathbb{A}$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ (\text{Var}) \qquad \frac{\Gamma, x : \sigma \vdash M : A}{\Gamma \vdash \lambda x.M : \sigma \to A}\ (\to\text{I})$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}\ (\to\text{E})$$

$$\frac{\Gamma \vdash M : \tau_1 \qquad \Gamma \vdash M : \tau_2}{\Gamma \vdash M : \tau_1 \cap \tau_2}\ (\cap\text{I}) \qquad \frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_i}\ (\cap\text{E})$$

Bounding Principles

Pervasive principles

- Functional *order* and *rank* of a logical operator (after Leivant [Lei83])

For intersection types:

- *Refinement*
  - After Freeman and Pfenning , PLDI 1991 [FP91]
- *Semantic types*
  - Bounded combinatory logic, TLCA 2011, CSL 2012 ff. [RU11; Düd+12]
- *Non-idempotence*
  - Bucciareli, Kesner, Ronchi Della Rocca, TCS 2014 [BKRDR14]
- *Norm and dimension*
  - D&R, POPL 2017, LICS 2017 [DR17a; DR17b]

λ-Calculus with Intersection Types: Inhabitation

- Inhabitation in λ-calculus with intersection types is undecidable
  - P. Urzyczyn, *The Emptiness Problem for Intersection Types*, JSL 1999 [Urz99] via reduction from halting problems for queue automata using rank 4 intersections
  - Related to the λ-definability problem
    - S. Salvati, *Recognizability in the Simply Typed Lambda-Calculus*, WoLLIC 2009 [Sal09]
    - Salvati, Manzonetto, Gehrke and Barendregt, *Urzyczyn and Loader are logically related*, ICALP 2012 [Sal+12]
    - Undecidability of λ-definability: Loader 1993 [Loa01]

Rank

Rank-bound wrt. a logical operator (after Leivant [Lei83]):

- Rank = maximal order (nesting depth to the left of $\rightarrow$) at which the operator can appear.
- Example: system $\mathbf{F}$ (operator $\forall$)
- Example: intersection types (operator $\cap$)

Definition (Intersection type rank)

$$\begin{aligned}
\mathbf{rank}(\tau) &= \mathbf{0} \text{ if } \tau \text{ is a simple type} \\
\mathbf{rank}(\sigma \rightarrow \tau) &= \mathbf{max}(\mathbf{rank}(\sigma) + \mathbf{1}, \mathbf{rank}(\tau)) \\
\mathbf{rank}(\sigma \cap \tau) &= \mathbf{max}(\mathbf{1}, \mathbf{rank}(\sigma), \mathbf{rank}(\tau))
\end{aligned}$$

λ-Calculus with Intersection Types: Rank-Bounded Inhabitation

Rank $\mathbf{2}$-inhabitation is decidable and Expspace-complete, and rank $\boldsymbol{k}$-inhabitation is undecidable for all ranks $\boldsymbol{k} > \mathbf{2}$

- Exptime-hard: D. Kuśmierek, *The Inhabitation Problem for Rank Two Intersection Types*, TLCA 2007 [Kus07]
- P. Urzyczyn, *Inhabitation of Low-Rank Intersection Types*, TLCA 2009 [Urz09]
- Proof techniques via *bus machines*, an alternating, expanding instruction device, also used to show Expxpace-completeness of inhabitation with explicit intersection [RU12]

λ-Calculus with Intersection Types: Dimensional bound

Inhabitation is decidable and Expspace-complete in every fixed multiset dimension. Generalization of decidable inhabitation under dimensional bound across all ranks.

- A. Dudenhefner & J.R., *Intersection Type Calculi of Bounded Dimension*, POPL 2017 [DR17a]

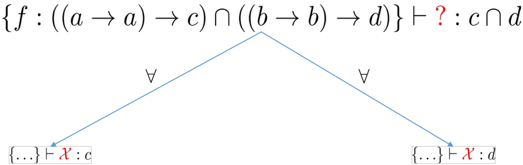
Typability is decidable and Pspace-complete in every fixed dimension.

- A. Dudenhefner & J.R., *Typability in Bounded Dimension*, LICS 2017 [DR17b]

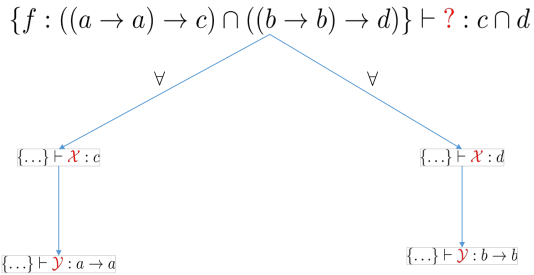
Intersection Type Inhabitation

$$\{f : ((a \to a) \to c) \cap ((b \to b) \to d)\} \vdash \textcolor{red}{?} : c \cap d$$

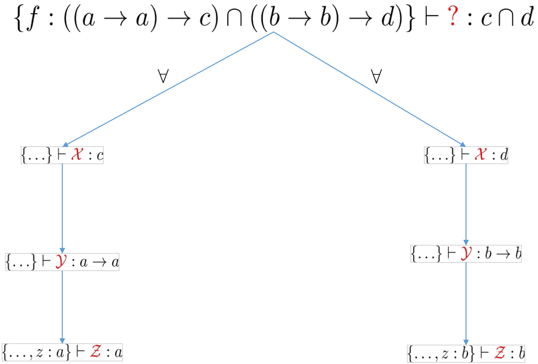
Intersection Type Inhabitation



Intersection Type Inhabitation



Intersection Type Inhabitation



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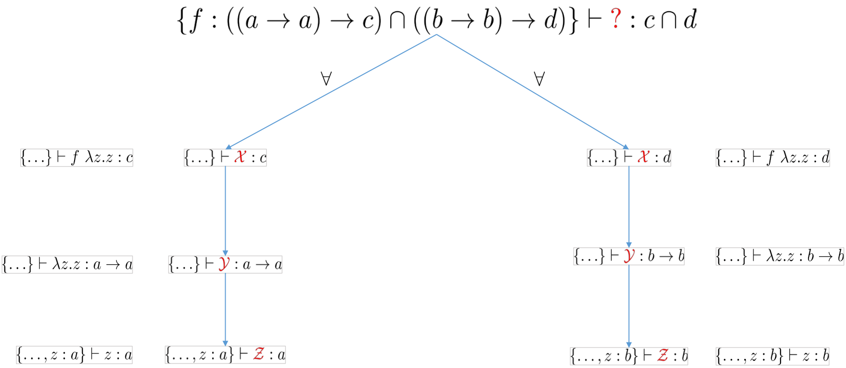

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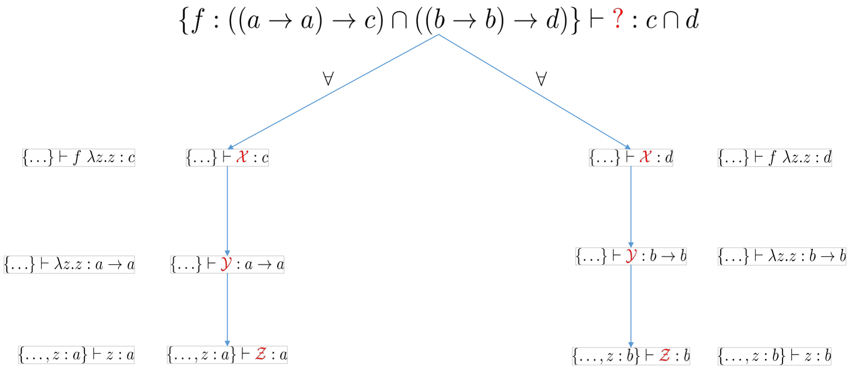


- Simultaneous constraints are *coupled* through $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$

Intersection Type Inhabitation

$$\Gamma = \{x : (e \cap f) \to g, y : (((a \to a) \cap (b \to b)) \to e) \cap (((b \to b) \cap (c \to c)) \to f)\}$$

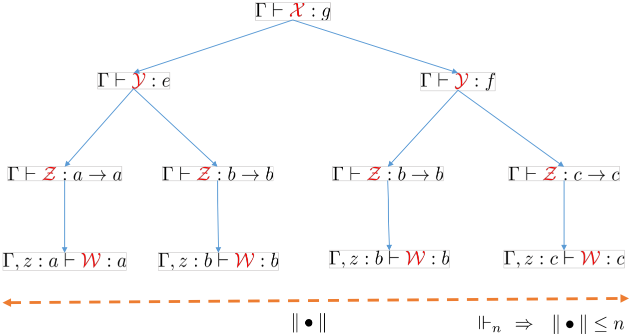
$$\Gamma \Vdash_n ? : g$$

$\mathcal{X} \equiv x(y(\lambda z.z))$

$\mathcal{Y} \equiv y(\lambda z.z)$

$\mathcal{Z} \equiv \lambda z.z$

$\mathcal{W} \equiv z$



Intersection Type Inhabitation

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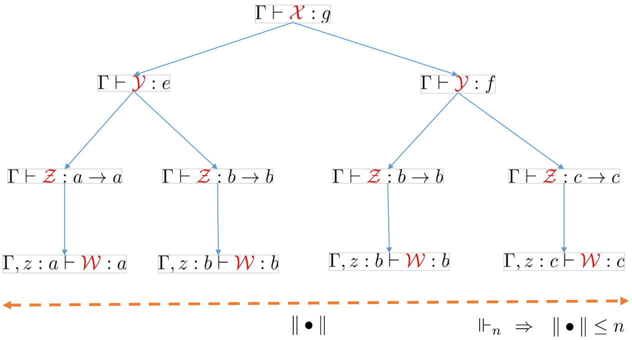

- See also our talk at TYPES 2016, *Rank* **3** *Inhabitation of Intersection Types Revisited* [Bes+16] and extended version at arXiv.

Combinatory Logic with Intersection Types $\textsc{cl}(\to, \cap)$

$$\frac{}{\Gamma, X : \tau \vdash X : S(\tau)}\text{(var)} \qquad \frac{\Gamma \vdash e : \tau \to \sigma \quad \Gamma \vdash e' : \tau}{\Gamma \vdash (e\, e') : \sigma}(\to\text{E})$$

$$\frac{\Gamma \vdash e : \tau \quad \Gamma \vdash e : \sigma}{\Gamma \vdash e : \tau \cap \sigma}(\cap\text{I}) \qquad \frac{\Gamma \vdash e : \tau \quad \tau \leq \sigma}{\Gamma \vdash e : \sigma}(\leq)$$

- Types are taken modulo associativity, commutativity and idempotence of $\cap$

Combinatory Logic with Intersection Types CL(→, ∩)

$$\frac{}{\Gamma, X : \tau \vdash X : S(\tau)}\text{(var)} \qquad \frac{\Gamma \vdash e : \tau \to \sigma \quad \Gamma \vdash e' : \tau}{\Gamma \vdash (e\, e') : \sigma}(\to\text{E})$$

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- Types are taken modulo associativity, commutativity and idempotence of ∩
- The subtyping relation ≤ is least preorder containing (cf. [BCDC83])

$$\sigma \leq \omega, \quad \omega \leq \omega \to \omega, \quad \sigma \cap \tau \leq \sigma, \quad \sigma \cap \tau \leq \tau, \quad \sigma \leq \sigma \cap \sigma;$$
$$(\sigma \to \tau) \cap (\sigma \to \rho) \leq \sigma \to \tau \cap \rho;$$
$$\text{If } \sigma \leq \sigma' \text{ and } \tau \leq \tau' \text{ then } \sigma \cap \tau \leq \sigma' \cap \tau' \text{ and } \sigma' \to \tau \leq \sigma \to \tau'.$$

- The **SKI**-calculus has been studied with intersection types (Dezani and Hindley [DCH92])

Note

But, in CLS, the combinatory theory Γ represents an arbitrary repository (basis not *fixed*)

Bounded Combinatory Logic $\text{BCL}_k(\to, \cap)$

- Levels

$$\begin{aligned}
\ell(a) &= 0, \text{ for } a \in \mathbb{A};\\
\ell(\tau \to \sigma) &= 1 + \max\{\ell(\tau), \ell(\sigma)\};\\
\ell(\textstyle\bigcap_{i=1}^{n} \tau_i) &= \max\{\ell(\tau_i) \mid i = 1, \ldots, n\}.\\
\\
\ell(S) &= \max\{\ell(S(\alpha)) \mid S(\alpha) \neq \alpha\}
\end{aligned}$$

Bounded Combinatory Logic $\text{BCL}_k(\to, \cap)$

- Levels

$$
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\\
\ell(S) &= \max\{\ell(S(\alpha)) \mid S(\alpha) \neq \alpha\}
\end{aligned}
$$

- $\text{BCL}_k(\to, \cap)$, $k \geq 0$ (and finite CL, FCL, with $S = id$).

$$
\frac{[\ell(S) \leq k]}{\Gamma, X : \tau \vdash_k X : S(\tau)}(\text{var}) \qquad \frac{\Gamma \vdash_k e : \tau \to \sigma \quad \Gamma \vdash_k e' : \tau}{\Gamma \vdash_k (e\, e') : \sigma}(\to\text{E})
$$

$$
\frac{\Gamma \vdash_k e : \tau \quad \Gamma \vdash_k e : \sigma}{\Gamma \vdash_k e : \tau \cap \sigma}(\cap\text{I}) \qquad \frac{\Gamma \vdash_k e : \tau \quad \tau \leq \sigma}{\Gamma \vdash_k e : \sigma}(\leq)
$$

- *Bounded Combinatory Logic*, CSL 2012 [Düd+12]

Complexity for Finite and Bounded CL

Theorem (TLCA 2011 [RU11])

For finite combinatory logic FCL:

1. *Relativized inhabitation in* FCL$(\to)$ *is in* PTIME
2. *Relativized inhabitation in* FCL$(\to, \cap)$ *is* EXPTIME*-complete*

Theorem (CSL 2012 [Düd+12])

For bounded combinatory logic BCL_k*:*

1. *Relativized inhabitation in* $\text{BCL}_k(\to)$ *is* EXPTIME*-complete for all* k
2. *Relativized inhabitation in* $\text{BCL}_k(\to, \cap)$ *is* $(k + 2)$-EXPTIME*-complete*

Upper Bound ATM for $\text{BCL}_k(\rightarrow, \cap)$: $\text{ASPACE}(\mathbf{exp}_{k+1}(n))$

$\textit{Input}: \quad \Gamma, \tau, k$

$\Gamma = \{f : (0 \to 1) \cap (1 \to 0),$
$\quad x : (\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma)\}$
$\tau = (0 \to 0) \cap (1 \to 1)$

Upper Bound ATM for $\textsc{bcl}_k(\to, \cap)$: $\textsc{Aspace}(\mathbf{exp}_{k+1}(n))$

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$\tau = (0 \to 0) \cap (1 \to 1)$

$\textit{loop}:$

CHOOSE $(x : \sigma) \in \Gamma$;
$\sigma' := \bigcap\{S(\sigma) \mid S \in \mathcal{S}_x^{(\Gamma,\tau,k)}\}$;

$\sigma' = (0 \to 0) \to (0 \to 0) \to (0 \to 0) \cap \cdots \cap$
$\quad (1 \to 1) \to (1 \to 1) \to (1 \to 1)$

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CHOOSE $m \in \{0, \ldots, \|\sigma'\|\}$;

CHOOSE $P \subseteq \mathbb{P}_m(\sigma')$;

$(0 \rightarrow 1) \rightarrow (1 \rightarrow 0) \rightarrow (0 \rightarrow 0) \cap$
$(1 \rightarrow 0) \rightarrow (0 \rightarrow 1) \rightarrow (1 \rightarrow 1)$

Upper Bound ATM for $\text{BCL}_k(\to, \cap)$: $\text{ASPACE}(\exp_{k+1}(n))$

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CHOOSE $m \in \{0, \ldots, \|\sigma'\|\}$;

CHOOSE $P \subseteq \mathbb{P}_m(\sigma')$;

IF $(\bigcap_{\pi \in P} tgt_m(\pi) \leq \tau)$ THEN

  IF $(m = 0)$ THEN ACCEPT;

$\Gamma = \{f : (0 \to 1) \cap (1 \to 0),$
$\quad x : (\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma)\}$
$\tau = (0 \to 0) \cap (1 \to 1)$

$\sigma' = (0 \to 0) \to (0 \to 0) \to (0 \to 0) \cap \cdots \cap$
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$(0 \to 1) \to (1 \to 0) \to (0 \to 0) \cap$
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$(0 \to 0) \cap (1 \to 1) \leq \tau$

Upper Bound ATM for $\text{BCL}_k(\rightarrow, \cap)$: $\text{ASPACE}(\exp_{k+1}(n))$

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loop :

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$\sigma' := \bigcap\{S(\sigma) \mid S \in \mathcal{S}_x^{(\Gamma,\tau,k)}\}$;

$\sigma' = (0 \rightarrow 0) \rightarrow (0 \rightarrow 0) \rightarrow (0 \rightarrow 0) \cap \cdots \cap$
$(1 \rightarrow 1) \rightarrow (1 \rightarrow 1) \rightarrow (1 \rightarrow 1)$

CHOOSE $m \in \{0, \ldots, ||\sigma'||\}$;
CHOOSE $P \subseteq \mathbb{P}_m(\sigma')$;

$(0 \rightarrow 1) \rightarrow (1 \rightarrow 0) \rightarrow (0 \rightarrow 0) \cap$
$(1 \rightarrow 0) \rightarrow (0 \rightarrow 1) \rightarrow (1 \rightarrow 1)$

IF $(\bigcap_{\pi \in P} tgt_m(\pi) \leq \tau)$ THEN

$(0 \rightarrow 0) \cap (1 \rightarrow 1) \leq \tau$

    IF $(m = 0)$ THEN ACCEPT;
    ELSE
        FORALL$(i = 1 \ldots m)$
            $\tau := \bigcap_{\pi \in P} arg_i(\pi)$;

$\tau := (0 \rightarrow 1) \cap (1 \rightarrow 0)$
$\tau := (1 \rightarrow 0) \cap (0 \rightarrow 1)$

        GOTO *loop*;
ELSE REJECT;

Upper Bound ATM for $\text{BCL}_k(\to, \cap)$: $\text{ASPACE}(\exp_{k+1}(n))$

$\mathit{Input}:\quad \Gamma, \tau, k$

$\mathit{loop}:$

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CHOOSE $m \in \{0, \ldots, \|\sigma'\|\}$;

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IF $(\bigcap_{\pi \in P} \mathit{tgt}_m(\pi) \leq \tau)$ THEN

  IF $(m = 0)$ THEN ACCEPT;

  ELSE

    FORALL$(i = 1 \ldots m)$

      $\tau := \bigcap_{\pi \in P} \mathit{arg}_i(\pi)$;

    GOTO loop;

ELSE REJECT;

$\Gamma = \{f : (0 \to 1) \cap (1 \to 0),$
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$\tau = (0 \to 0) \cap (1 \to 1)$

$\sigma' = (0 \to 0) \to (0 \to 0) \to (0 \to 0) \cap \cdots \cap$
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$(0 \to 1) \to (1 \to 0) \to (0 \to 0) \cap$
$(1 \to 0) \to (0 \to 1) \to (1 \to 1)$

$(0 \to 0) \cap (1 \to 1) \leq \tau$

$\tau := (0 \to 1) \cap (1 \to 0)$
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$(x\ f)\ f : (0 \to 0) \cap (1 \to 1)$

Lower bound - main ideas

- Generic reduction by simulation of $\mathbf{exp}_{k+1}(n)$-space bounded alternating Turing machines.
- Given ATM $\mathcal{M}$, we construct, in polynomial time, an environment Γ such that $\mathcal{M}$ is accepting if and only if $\Gamma \vdash_k ? : \textbf{Tape}$ is solvable

Lower bound - main ideas

- Generic reduction by simulation of $\mathbf{exp}_{k+1}(n)$-space bounded alternating Turing machines.
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For any fixed level-parameter K:

- Intersection type numerals $\langle i \rangle_K$: we can represent numbers $0 \leq i \leq \mathbf{exp}_{K+1}(n) - 1$ as intersection types.
- We can represent ATM configurations $\mathscr{C}$ of size $\mathbf{exp}_{K+1}(n)$ as intersection types $[\mathscr{C}]$
- Exploiting K-bounded polymorphism, we can represent these types *implicitly* in polynomial sized types
- ATM sequences $\mathscr{C}_1 \mathscr{C}_2 \cdots \mathscr{C}_m$ coded by reverse implications $[\mathscr{C}_{i+1}] \to [\mathscr{C}_i]$ in Γ

Lower bound: intersection type numerals

$\mathbb{B} = \{\mathbf{0}, \mathbf{1}\}$, $\mathbb{B}[n] = \{\mathbf{0}_1, \ldots, \mathbf{0}_n\} \cup \{\mathbf{1}_1, \ldots, \mathbf{1}_n\}$

- $\mathcal{N}_0 = \{\bigcap_{i=1}^{n} \mathfrak{b}_i \mid \mathfrak{b}_i \in \{\mathbf{0}_i, \mathbf{1}_i\}$ for $i = 1 \ldots n\}$
- $\mathcal{N}_{k+1} = \{\bigcap_{\tau \in \mathcal{N}_k} (\tau \to b_\tau) \mid b_\tau \in \{\mathbf{0}, \mathbf{1}\},$ for $\tau \in \mathcal{N}_k\}$

Lower bound: intersection type numerals

$\mathbb{B} = \{0, 1\}$, $\mathbb{B}[n] = \{0_1, \ldots, 0_n\} \cup \{1_1, \ldots, 1_n\}$

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- $[\![\bigcap_{i=1}^{n} \mathfrak{b}_i]\!] = \sum_{i=1}^{n} [\![\mathfrak{b}_i]\!] \times 2^{i-1}$, with $[\![0_i]\!] = 0$ and $[\![1_i]\!] = 1$
- $[\![\bigcap_{\sigma \in \mathcal{N}_k} (\tau \to b_\tau)]\!] = \sum_{\tau \in \mathcal{N}_k} b_\tau \times 2^{[\![\tau]\!]}$

- $[\![0_1 \cap 1_2 \cap 0_3 \cap 1_4]\!] = 2 + 8 = 10$
- $[\![((0_1 \cap 0_2) \to 0) \cap ((0_1 \cap 1_2) \to 1) \cap ((1_1 \cap 0_2) \to 0) \cap ((1_1 \cap 1_2) \to 1)]\!] = 10$

Lower bound: intersection type numerals

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- $[\![((0_1 \cap 0_2) \to 0) \cap ((0_1 \cap 1_2) \to 1) \cap ((1_1 \cap 0_2) \to 0) \cap ((1_1 \cap 1_2) \to 1)]\!] = 10$

- $\sigma \in \mathcal{N}_k$: $0 \leq [\![\sigma]\!] \leq \exp_{k+1}(n) - 1$, uniquely

Configurations

The explicit representation of an ATM configuration $\mathscr{C}$ is

$$[\mathscr{C}] = \bigcap_{i=0}^{\exp_{K+1}(n)-1} \textbf{Cell}(a_i, q, \langle m \rangle_K, \langle i \rangle_K)$$

where

- $\langle i \rangle_K$ is the address of the cell
- $\langle m \rangle_K$ is the address of the cell under the r/w-head
- q is the control state
- a_i is the symbol in the cell

Configurations

The explicit representation of an ATM configuration $\mathscr{C}$ is

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where

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- q is the control state
- a_i is the symbol in the cell

We will represent configurations *implicitly* as $\textbf{Cell}(\alpha, \beta, \gamma, \delta)$

Refinement (after [FP91])

Definition ([Sal+12])

Let $\mathbb{T}_o$ be simple types over an atom o. Fix a finite set $X \subseteq \mathbb{A}$ and define *uniform types* $\mathcal{U}_X(\tau)$ for $\tau \in \mathbb{T}_o$:

$$\begin{aligned} \mathcal{U}_X(o) &= X^{\cap} \\ \mathcal{U}_X(\tau \to \sigma) &= (\mathcal{U}_X(\tau) \Rightarrow \mathcal{U}_X(\sigma))^{\cap} \end{aligned}$$

where

$$\begin{aligned} Y^{\cap} &= \{\sigma_1 \cap \cdots \cap \sigma_n \mid \sigma_i \in Y \text{ for } i = 1 \ldots n\} \\ Y \Rightarrow Z &= \{\sigma \to \tau \mid \sigma \in Y, \tau \in Z\} \end{aligned}$$

for sets Y, Z of intersection types.

Refinement (after [FP91])

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$$\begin{array}{lcl} \mathcal{U}_X(o) & = & X^{\cap} \\ \mathcal{U}_X(\tau \to \sigma) & = & (\mathcal{U}_X(\tau) \Rightarrow \mathcal{U}_X(\sigma))^{\cap} \end{array}$$

- With such types we can represent any finite function $f : \mathbf{A} \to \mathbf{B}$ at the type level by $\bigcap_{a \in \mathbf{A}} (a \to f(a))$

Refinement (after [FP91])

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$$\begin{array}{lcl} \mathtt{succ} & : & (\mathtt{Nat} \to \mathtt{Nat}) \cap (\mathtt{zero} \to \mathtt{pos}) \cap (\mathtt{pos} \to \mathtt{pos}) \cap \\ & & (\mathtt{even} \to \mathtt{odd}) \cap (\mathtt{odd} \to \mathtt{even}) \end{array}$$

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- *Inhabitation* (λ-calculus) is undecidable.

Refinement (after [FP91])

Definition

Let $\mathbb{T}_o$ be simple types over an atom o. Fix $X \subseteq \mathbb{A}$ and define *uniform types* $\mathcal{U}_X(\tau)$ for $\tau \in \mathbb{T}_o$:

$$
\begin{aligned}
\mathcal{U}_X(o) &= X^{\cap} \\
\mathcal{U}_X(\tau \to \sigma) &= (\mathcal{U}_X(\tau) \Rightarrow \mathcal{U}_X(\sigma))^{\cap}
\end{aligned}
$$

- With such types we can represent any finite function $f : \mathbf{A} \to \mathbf{B}$ at the type level by $\bigcap_{a \in \mathbf{A}}(a \to f(a))$
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$$
\begin{aligned}
\mathtt{succ} \quad : \quad & (\mathtt{Nat} \to \mathtt{Nat}) \cap (\mathtt{zero} \to \mathtt{pos}) \cap (\mathtt{pos} \to \mathtt{pos}) \cap \\
& (\mathtt{even} \to \mathtt{odd}) \cap (\mathtt{odd} \to \mathtt{even})
\end{aligned}
$$

- *Inhabitation* (λ-calculus) is undecidable. *Proof*: Note that [Sal+12] uses only uniform types for λ-definability. □

CL(→, ∩) over Uniform (Refinement) Types

Definition

Let $\mathbb{T}_o$ be simple types over an atom o. Fix $X \subseteq \mathbb{A}$ and define *uniform types* $\mathcal{U}_X(\tau)$ for $\tau \in \mathbb{T}_o$:

$$\begin{array}{lcl} \mathcal{U}_X(o) & = & X^\cap \\ \mathcal{U}_X(\tau \to \sigma) & = & (\mathcal{U}_X(\tau) \Rightarrow \mathcal{U}_X(\sigma))^\cap \end{array}$$

Corollary

Relativized inhabitation with uniform types is nonelementary recursive.

Proof.

Upper bound: every problem $\Gamma \vdash ? : \sigma$ is decidable within $\text{BCL}_k(\to, \cap)$ with $k = \max\{\ell(\tau) \mid \tau \in rn(\Gamma)\}$.
Lower bound: notice that all constructions in l.b. for $\text{BCL}_k(\to, \cap)$ can be carried out with uniform types. □

Corollary: Henkin's theory Ω in $\mathrm{BCL}_k(\to, \cap)$

Satisfiability of formulae

$$\Phi ::= 0 \in x^1 \mid 1 \in x^1 \mid x^k \in y^{k+1} \mid \neg\Phi \mid \forall x^k.\Phi \mid \Phi \wedge \Phi'$$

where x^k ranges over $\mathbf{D}_k$ with $\mathbf{D}_0 = \{0, 1\}$, $\mathbf{D}_{k+1} = \mathcal{P}(\mathbf{D}_k)$.

L. Henkin: A theory of propositional types, Fundamenta Informaticae 52 (1963) 323–344.

Representation in $\mathrm{BCL}_k(\to, \cap)$ (for sufficiently large k):

- A set variable x^k is represented by a type variable x^k.
- Membership predicate Mem_k

  $$Num_k(x^k) \to Num_{k+1}(y^{k+1}) \to In_k(x^k, y^{k+1}) \to Mem_k(x^k, y^{k+1})$$

  where $In_k(x^k, x^k \to 1)$ and $NotIn(x^k, x^k \to 0)$ are axioms.
- Use alternation to code quantifiers as usual (Urzyczyn 1997).

Ongoing: optimization & algorithm engineering

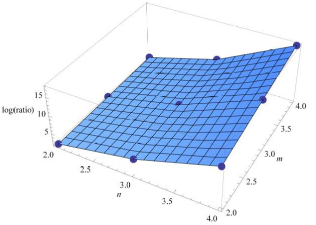
From B. Düdder: *Automatic Synthesis of Component & Connector-Software Architectures with Bounded Combinatory Logic*, Diss. Dortmund, Aug. 2014.

Algorithm 4.5: ATM with lookahead-test

Input : Γ, τ — all types in Γ and $\tau = \bigcap_{i \in I} \tau_i$ organized
loop :
CHOOSE $(x : \sigma) \in \Gamma$;
write $\sigma \equiv \bigcap_{j \in J} \sigma_j$
FOR EACH $i \in I, j \in J, m \leq \|\sigma\|$ DO
    $\texttt{candidates}(i, j, m) := \texttt{Match}(tgt_m(\sigma_j) \leq \tau_i)$
$M := \{m \leq \|\sigma\| \mid \forall_{i \in I} \exists_{j \in J} : \texttt{candidates}(i, j, m) = \textbf{true}\}$
CHOOSE $m \in M$;
FOR EACH $i \in I$ DO
    CHOOSE $j_i \in J$ with $\texttt{candidates}(i, j_i, m) = \textbf{true}$
    CHOOSE S_i a substitution
    CHOOSE $\pi_i \in \mathbb{P}_m(\overline{S_i(\sigma_{j_i})})$ with $tgt_m(\pi_i) \leq \tau_i$ and
    $\forall 1 \leq l \leq m \; \forall \pi' \in \overline{arg_l(\pi_i)} \; \exists (y : \rho) \in \Gamma \; \exists$ a path ρ'
    in $\rho \; \exists k$: $\texttt{Match}(tgt_k(\rho') \leq \pi') = \textbf{true}$
IF $(m = 0)$ THEN ACCEPT;
ELSE FORALL$(l = 1 \ldots m)$
        $\tau := \bigcap_{i \in I} arg_l(\pi_i)$;
        GOTO *loop*;

		Unoptimized ATM		Lookahead-ATM	
n	m	#I	$\overline{\text{RT}}$/ms	#I	$\overline{\text{RT}}$/ms
2	2	73	112.2	9	109.8
2	3	973	209.6	49	111.4
2	4	11 665	2245.2	257	124.4
3	2	43 905	12 504	55	124
3	3	$1.4 \times 10^{10*}$	–	2188	354
3	4	$4.8 \times 10^{12*}$	–	78 733	50 959.6
4	2	$1.3 \times 10^{14*}$	–	33	142.4
4	3	$6.6 \times 10^{18*}$	–	3889	1314.6
4	4	$3.3 \times 10^{23*}$	–	640 001	7.5×10^6

Table 4.1: Experimental Data for Γ_n^m



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