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Exercise 2: Subtyping

Given the following rules:

Type assignment rules for combinatory logic with intersection types and distributing covariant constructors.

$n \in \mathbb{N}^0$	Natural number or zero
$x, y, z \in \mathbb{O}$	Combinators
$\alpha, \beta, \gamma \in \mathbb{V}$	Type Variables
$C, C_1, C_2, \dots, C_n \in \mathbb{C}$	Constructor Symbols
$\lvert \cdot \rvert : \mathbb{C} \to \mathbb{N}^0$	Constructor Arity
$\leq_{\mathbb{C}} \subseteq \{(C_1, C_2) \in \mathbb{C} \times \mathbb{C} \mid \lvert C_1 \rvert = \lvert C_2 \rvert\}$	Constructor Preorder
$S : \mathbb{V} \to \mathbf{T}$	Substitution
$S^*(\sigma) =$	S lifted to $\mathbb{T}_{\mathbb{V}} \to \mathbf{T}$
$C(S^*(\sigma_1), S^*(\sigma_2), \dots, S^*(\sigma_{\lvert C \rvert}))$	if $\tau = C(\sigma_1, \sigma_2, \dots, \sigma_{\lvert C \rvert})$
ω	if $\tau = \omega$
$S^*(\sigma) \to S^*(\tau)$	if $\tau = \sigma \to \tau$
$S^*(\sigma) \cap S^*(\tau)$	if $\tau = \sigma \cap \tau$
$S(\alpha)$	if $\tau = \alpha$

$\mathbb{T}_{\mathbb{V}} \ni \sigma, \tau, \rho, \sigma_1, \sigma_2, \dots, \sigma_n ::=$	Type Scheme
$C(\sigma_1, \sigma_2, \dots, \sigma_{\lvert C \rvert})$	constant
ω	omega
$\sigma \to \tau$	function
$\sigma \cap \tau$	intersection
α	variable of $\mathbb{V}$
$\mathbf{T} = \mathbb{T}_{\emptyset} \ni \mathtt{a}, \mathtt{a}_1, \mathtt{a}_2, \dots, \mathtt{a}_n, \mathtt{b}, \mathtt{b}_1, \mathtt{b}_2, \dots, \mathtt{b}_n$	Type
$\mathcal{T} \ni M, N, N_1, N_2, \dots, N_n ::=$	Term
x	variable
$(M\ N)$	application
$\Gamma : \mathbb{O} \to \mathbb{T}_{\mathbb{V}}$	Type Context
$\mathrm{WF} \subseteq \mathbb{V} \to \mathbf{T}$	Function space of well-formed substitutions

Subtyping ($\leq \subseteq \mathbf{T} \times \mathbf{T}$):

$$\frac{C_1 \leq_{\mathbb{C}} C_2 \qquad \lvert C_1 \rvert = \lvert C_2 \rvert \qquad \forall n : \mathtt{a}_n \leq \mathtt{b}_n}{C_1(\mathtt{a}_1, \mathtt{a}_2, \dots, \mathtt{a}_{\lvert C_1 \rvert}) \leq C_2(\mathtt{b}_1, \mathtt{b}_2, \dots, \mathtt{b}_{\lvert C_2 \rvert})}\ (\text{CAx}) \qquad \frac{}{\mathtt{a} \leq \omega}\ (\leq \omega)$$

$$\frac{}{C_1(\mathtt{a}_1, \mathtt{a}_2, \dots, \mathtt{a}_{\lvert C_1 \rvert}) \cap C_1(\mathtt{b}_1, \mathtt{b}_2, \dots, \mathtt{b}_{\lvert C_1 \rvert}) \leq C_1(\mathtt{a}_1 \cap \mathtt{b}_1, \mathtt{a}_2 \cap \mathtt{b}_2, \dots, \mathtt{a}_{\lvert C_1 \rvert} \cap \mathtt{b}_{\lvert C_1 \rvert})}\ (\text{CDist})$$

$$\frac{}{\omega \leq \omega \to \omega}\ (\to \omega) \qquad \frac{\mathtt{a}_2 \leq \mathtt{a}_1 \qquad \mathtt{b}_1 \leq \mathtt{b}_2}{\mathtt{a}_1 \to \mathtt{b}_1 \leq \mathtt{a}_2 \to \mathtt{b}_2}\ (\text{Sub}) \qquad \frac{}{(\mathtt{a} \to \mathtt{b}_1) \cap (\mathtt{a} \to \mathtt{b}_2) \leq \mathtt{a} \to \mathtt{b}_1 \cap \mathtt{b}_2}\ (\text{Dist})$$

$$\frac{}{\mathtt{a} \leq \mathtt{a} \cap \mathtt{a}}\ (\text{Idem}) \quad \frac{\mathtt{a}_1 \leq \mathtt{a}_2 \qquad \mathtt{a}_2 \leq \mathtt{a}_3}{\mathtt{a}_1 \leq \mathtt{a}_3}\ (\text{Trans}) \quad \frac{\mathtt{a} \leq \mathtt{b}_1 \qquad \mathtt{a} \leq \mathtt{b}_2}{\mathtt{a} \leq \mathtt{b}_1 \cap \mathtt{b}_2}\ (\text{GLB}) \quad \frac{}{\mathtt{a} \cap \mathtt{b} \leq \mathtt{a}}\ (\text{LUB}_1) \quad \frac{}{\mathtt{a} \cap \mathtt{b} \leq \mathtt{b}}\ (\text{LUB}_2)$$

Type assignment ($\vdash \subseteq (\mathbb{O} \to \mathbb{T}_{\mathbb{V}}) \times \mathcal{T} \times \mathbf{T}$):

$$\frac{\Gamma(x) = \tau \qquad S \in \mathrm{WF}}{\Gamma \vdash x : S^*(\tau)}\ (\text{Var}) \qquad \frac{\Gamma \vdash M : \mathtt{a} \to \mathtt{b} \qquad \Gamma \vdash N : \mathtt{a}}{\Gamma \vdash (M\ N) : \mathtt{b}}\ (\to \text{E})$$

$$\frac{\Gamma \vdash M : \mathtt{a} \qquad \mathtt{a} \leq \mathtt{b}}{\Gamma \vdash M : \mathtt{b}}\ (\leq) \qquad \frac{\Gamma \vdash M : \mathtt{a} \qquad \Gamma \vdash M : \mathtt{b}}{\Gamma \vdash M : \mathtt{a} \cap \mathtt{b}}\ (\cap \text{I})$$

Show that the following statements apply to all types from $\mathbf{T}$

1. $\mathtt{a} \leq \mathtt{a}$
2. $\omega \leq \mathtt{a} \to \omega$
3. $(\mathtt{a}_1 \to \mathtt{b}_1) \cap (\mathtt{a}_2 \to \mathtt{b}_2) \leq (\mathtt{a}_1 \cap \mathtt{a}_2) \to (\mathtt{b}_1 \cap \mathtt{b}_2)$
4. $\mathtt{a} \cap \mathtt{b} \leq \mathtt{b} \cap \mathtt{a}$
5. if $\mathtt{a}_1 \leq \mathtt{a}_2$ and $\mathtt{b}_1 \leq \mathtt{b}_2$, then $\mathtt{a}_1 \cap \mathtt{b}_1 \leq \mathtt{a}_2 \cap \mathtt{b}_2$

Exercise 3: Type derivations

Find a term and its derivations for the following problems

1.

$$\begin{aligned}
&\mathrm{WF} = \{\{\alpha \mapsto \mathit{DropDown}\}, \{\alpha \mapsto \mathit{RadioButtons}\}\}\\
&\Gamma = \{ \quad \texttt{customerForm} : (\texttt{String} \to \texttt{java.net.URL} \to \texttt{OptionSelection} \to \texttt{Form}) \cap\\
&\qquad (\mathit{Title} \to \mathit{Location}(\mathit{Logo}) \to \mathit{ChoiceDialog}(\alpha) \to \mathit{OrderMenu}(\alpha))\\
&\qquad \texttt{dropDownSelector} : (\texttt{java.net.URL} \to \texttt{OptionSelection}) \cap\\
&\qquad (\mathit{Location}(\mathit{Database}) \to \mathit{ChoiceDialog}(\mathit{DropDown}))\\
&\qquad \texttt{radioButtonSelector} : (\texttt{java.net.URL} \to \texttt{OptionSelection}) \cap\\
&\qquad (\mathit{Location}(\mathit{Database}) \to \mathit{ChoiceDialog}(\mathit{RadioButtons}))\\
&\qquad \texttt{companyTitle} : \texttt{String} \cap \mathit{Title}\\
&\qquad \texttt{databaseLocation} : \texttt{java.net.URL} \cap \mathit{Location}(\mathit{Database})\\
&\qquad \texttt{logoLocation} : \texttt{java.net.URL} \cap \mathit{Location}(\mathit{Logo})\\
&\qquad \texttt{alternateLogoLocation} : \texttt{java.net.URL} \cap \mathit{Location}(\mathit{Logo}) \quad \}\\
&\Gamma \vdash ? : \texttt{Form} \cap \mathit{OrderMenu}(\omega)
\end{aligned}$$

2. **(This exercise is optional!)**

$$\begin{aligned}
&\mathrm{WF} = \{\{\alpha \mapsto \mathit{blue}\}, \{\alpha \mapsto \mathit{red}\}\}\\
&\leq_{\mathbb{C}} = \{(\mathit{Shopper}, \mathit{Person})\}\\
&\Gamma = \{\texttt{hipster} : \mathit{Item}(\mathit{red} \cap \mathit{blue}) \to \mathit{Shopper} \cap \mathit{Happy},\ \texttt{Shoes} : \mathit{Item}(\alpha)\}\\
&\Gamma \vdash ? : \mathit{Person} \cap \mathit{Happy}
\end{aligned}$$