# Assignments for the Warsaw Open PhD Lecture *A Type-Based Approach to Component-oriented Synthesis*

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**Assignment 1 (Combinators)** *Show that the combinator* **I** *can be coded in terms of* **S** *and* **K**.

**Assignment 2 (Combinators)** *Schönfinkel's combinators* **B** *and* **C** *are given by the definitions:*

$$\begin{array}{lll} \mathbf{B}FGH & \rhd_B & F G H \\ \mathbf{C}FGH & \rhd_C & FHG \end{array}$$

1. *Prove that the combinators* **B** *and* **C** *can be expressed using only* **K** *and* **S**.
2. *Which are the most general types we can give to* **B** *and* **C** *in the simple type system?*

**Assignment 3 (Inhabitation)** *Describe an infinite set of distinct $\lambda$-terms in $\beta$-normal form inhabiting the type $(\alpha \to \alpha) \to (\alpha \to \alpha)$ in the simple type system for the $\lambda$ calculus.*

**Assignment 4 (Inhabitation)** *Consider the encoding of the two-counter automaton from Lecture 2. We have the following:*

THEOREM Let $\mathcal{A}$ be a two-counter automaton with initial configuration $(q_0, n_0, m_0)$. $\mathcal{A}$ accepts if and only if there exists a term $e$ with $\Gamma_{\mathcal{A}} \vdash e : q_0 \to s^{n_0}(0) \to s^{m_0}(0)$.

LEMMA 1 Let $\mathscr{C}$ and $\mathscr{C}'$ be configurations in $\mathcal{A}$. We have $\mathscr{C} \to \mathscr{C}'$ if and only if there is a term $e$ with $\Gamma_{\mathcal{A}} \vdash e : [\mathscr{C}'] \to [\mathscr{C}]$.

LEMMA 1 Let $\mathscr{C}$ be a configuration of $\mathcal{A}$. $\mathscr{C}$ leads to acceptance in $\mathcal{A}$ if and only if there is a term $e$ with $\Gamma_{\mathcal{A}} \vdash e : [\mathscr{C}]$.

*Prove* THEOREM *by proving* LEMMA 1 *and* LEMMA 2 *and showing how the theorem follows from them.*

**Assignment 5 (Intersection types)** *Refer to the intersection type system for the $\lambda$ calculus given in Lecture 2.*

*Let $\omega$ be the $\lambda$-term*

$$\omega = \lambda x.(x\ x)$$

*and let $\Omega$ be the $\lambda$-term*

$$\Omega = (\omega\ \omega)$$

1. *Show that $\omega$ has a type in the intersection type system.*

2. *Does $\Omega$ have a type in the system? Explain your answer.*

**Assignment 6 (Combinatory synthesis)** *Let $\Gamma$ be the following collection of combinators, where $\alpha, \beta, \gamma$ are type variables,* $[\_]$ *and* `Graph(_)` *are type constructors, and* `PartialOrder` *and* `TopSorted` *are semantic types:*

$$
\begin{array}{lcl}
\mathbf{F} & : & [\alpha] \to (\alpha \to \mathtt{bool}) \to [\alpha] \\
\mathbf{S} & : & ([\alpha] \to (\alpha \to \mathtt{bool}) \to [\alpha]) \to \\
 & & \quad ((\alpha \to \alpha \to \mathtt{bool}) \cap \mathtt{PartialOrder}) \to [\alpha] \to ([\alpha] \cap \mathit{TopSorted}) \\
\mathbf{G} & : & \mathit{Graph}(\alpha) \to ((\alpha \to \alpha \to \mathtt{bool}) \cap \mathtt{PartialOrder}) \\
\mathbf{N} & : & \mathit{Graph}(\alpha) \to [\alpha] \\
\circ & : & (\alpha \to \beta) \to (\beta \to \gamma) \to (\alpha \to \gamma) \\
\diamond & : & (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to (\alpha \to \gamma)
\end{array}
$$

*Show that the following inhabitation (synthesis) problem has a solution using combinatory logic with intersection types (Lecture 3):*

$$\Gamma \vdash ? : \mathit{Graph}(\alpha) \to ([\alpha] \cap \mathit{TopSorted})$$

*It is sufficient to exhibit an inhabitant solving the problem.*