Threesomes, with and without blame (POPL 2010)

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My whole life I had assumed
there were threesomes in Heaven.

Even and odd, untyped

$$
\ulcorner \texttt{let}
$$

$$
\begin{aligned}
&\quad \mathit{even}^\star = \lambda n.\, \texttt{if}\ n = 0 \\
&\qquad\qquad\qquad\quad \texttt{then true} \\
&\qquad\qquad\qquad\quad \texttt{else}\ \mathit{odd}^\star\,(n-1) \\
&\quad \mathit{odd}^\star = \lambda n.\, \texttt{if}\ n = 0 \\
&\qquad\qquad\qquad\quad \texttt{then false} \\
&\qquad\qquad\qquad\quad \texttt{else}\ \mathit{even}^\star\,(n-1) \\
&\texttt{in} \\
&\quad \mathit{even}^\star\, 2 \urcorner
\end{aligned}
$$

Even and odd, partly typed

$$
\begin{array}{l}
\texttt{let} \\
\quad \mathit{even}^\star = \lceil \lambda n.\, \texttt{if}\ n = 0 \\
\qquad\qquad\qquad\quad \texttt{then true} \\
\qquad\qquad\qquad\quad \texttt{else}\ \mathit{odd}^\star\,(n-1) \rceil \\
\quad \mathit{odd} = \lambda n : \texttt{Int}.\, \texttt{if}\ n = 0 \\
\qquad\qquad\qquad\qquad \texttt{then false} \\
\qquad\qquad\qquad\qquad \texttt{else}\ \mathit{even}\,(n-1) \\
\quad \mathit{even} = \mathit{even}^\star : \star \overset{p}{\Rightarrow} \texttt{Int} \to \texttt{Int} \\
\quad \mathit{odd}^\star = \mathit{odd} : \texttt{Int} \to \texttt{Int} \overset{q}{\Rightarrow} \star \\
\texttt{in} \\
\quad \lceil \mathit{even}^\star\ 2 \rceil
\end{array}
$$

Goodbye tail recursion!

$$\lceil even^{\star}\ 4 \rceil$$

$$\longrightarrow$$

$$(odd\ 3) : \texttt{Bool} \overset{q}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$\lceil even^{\star}\ 2 \rceil : \star \overset{p}{\Longrightarrow} \texttt{Bool} \overset{q}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$(odd\ 1) : \texttt{Bool} \overset{q}{\Longrightarrow} \star \overset{p}{\Longrightarrow} \texttt{Bool} \overset{q}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$\lceil even^{\star}\ 0 \rceil : \star \overset{p}{\Longrightarrow} \texttt{Bool} \overset{q}{\Longrightarrow} \star \overset{p}{\Longrightarrow} \texttt{Bool} \overset{q}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$\lceil \texttt{true} \rceil$$

Recovering tail recursion!

$$\lceil \mathit{even}^{\star}\ 4 \rceil$$

$$\longrightarrow$$

$$(\mathit{odd}\ 3) : \texttt{Bool} \overset{q}{\Rightarrow} \star$$

$$\longrightarrow$$

$$\lceil \mathit{even}^{\star}\ 2 \rceil : \star \overset{p}{\Rightarrow} \texttt{Bool} \overset{q}{\Rightarrow} \star$$

$$\longrightarrow$$

$$(\mathit{odd}\ 1) : \texttt{Bool} \overset{q}{\Rightarrow} \star$$

$$\longrightarrow$$

$$\lceil \mathit{even}^{\star}\ 0 \rceil : \star \overset{p}{\Rightarrow} \texttt{Bool} \overset{q}{\Rightarrow} \star$$

$$\longrightarrow$$

$$\lceil \texttt{true} \rceil$$

Part I

Twosomes: a small change to blame calculus

Add the empty type

$$\text{Types} \quad A, B, C \quad ::= \quad \iota \mid A \to B \mid \star \mid \bot$$

Every type is incompatible with $\bot$.

$$A \not\prec \bot \qquad \bot \not\prec A$$

There are no values of type $\bot$.

Allow casts between any types

Change one type rule

$$\frac{\Gamma \vdash s : A \qquad A \prec B}{\Gamma \vdash (s : A \stackrel{p}{\Rightarrow} B) : B}$$

becomes

$$\frac{\Gamma \vdash s : A}{\Gamma \vdash (s : A \stackrel{p}{\Rightarrow} B) : B}$$

Add one reduction

$$v : A \stackrel{p}{\Rightarrow} \bot \quad \longrightarrow \quad \texttt{blame}\, p$$

Retain wrapping casts as values

$$\text{Values} \quad v, w \quad ::= \quad c \mid \lambda x{:}A.\, t \mid v : G \Rightarrow \star \mid$$
$$v : A \to B \stackrel{p}{\Rightarrow} A' \to B'$$

Change the wrap rule

$$v : A{\to}B \stackrel{p}{\Rightarrow} A'{\to}B' \quad \longrightarrow \quad \lambda x'{:}A'.\, (v\, (x' : A' \stackrel{\bar{p}}{\Rightarrow} A) : B \stackrel{p}{\Rightarrow} B')$$

becomes

$$(v : A{\to}B \stackrel{p}{\Rightarrow} A'{\to}B')\, w \quad \longrightarrow \quad v\, (w : A' \stackrel{\bar{p}}{\Rightarrow} A) : B \stackrel{p}{\Rightarrow} B'$$

Part II

Naive subtype and meet

Naive subtype

$$\frac{}{A <:_n \star}$$

$$\frac{}{\iota <:_n \iota}$$

$$\frac{A <:_n A' \qquad B <:_n B'}{A \to B <:_n A' \to B'}$$

Shallow incompatibility

$$A \# \bot \qquad \bot \# A$$

$$\iota \# \iota' \text{ if } \iota \neq \iota'$$

$$\iota \# A \to B \qquad A \to B \# \iota$$

A base type is shallowly compatible with itself.

Any two functions are shallowly compatible.

Everything is shallowly compatible with $\star$.

Meet

$$
\begin{aligned}
A \mathbin{\&} \star &= A \\
\star \mathbin{\&} A &= A \\
\iota \mathbin{\&} \iota &= \iota \\
A \to B \mathbin{\&} A' \to B' &= (A \mathbin{\&} A') \to (B \mathbin{\&} B') \\
A \mathbin{\&} B &= \bot \qquad \text{if } A \# B
\end{aligned}
$$

$A \mathbin{\&} B$ is the greatest lower bound of A and B.

$C <:_n A \mathbin{\&} B$ if and only if $C <:_n A$ and $C <:_n B$

First factoring theorem

$$v : A \stackrel{p}{\Longrightarrow} B \quad = \quad v : A \stackrel{p}{\Longrightarrow} A \,\&\, B \stackrel{p}{\Longrightarrow} B$$

An example

$$v : \texttt{Int} \longrightarrow \star \stackrel{p}{\Longrightarrow} \star \longrightarrow \texttt{Bool}$$

$$=$$

$$v : \texttt{Int} \longrightarrow \star \stackrel{p}{\Longrightarrow} \texttt{Int} \longrightarrow \texttt{Bool} \stackrel{p}{\Longrightarrow} \star \longrightarrow \texttt{Bool}$$

Second factoring theorem

$$v : A \stackrel{p}{\Longrightarrow} B \quad = \quad v : A \stackrel{p}{\Longrightarrow} \star \stackrel{p}{\Longrightarrow} B$$

An example

$$v : \texttt{Int} \longrightarrow \star \stackrel{p}{\Longrightarrow} \star \longrightarrow \texttt{Bool}$$

$$=$$

$$v : \texttt{Int} \longrightarrow \star \stackrel{p}{\Longrightarrow} \star \stackrel{p}{\Longrightarrow} \star \longrightarrow \texttt{Bool}$$

Third factoring theorem

$$v : A \overset{p}{\Rightarrow} B \quad = \quad v : A \overset{p}{\Rightarrow} C \overset{p}{\Rightarrow} B$$

$$\text{whenever } A \mathbin{\&} B <:_n C$$

This result subsumes the other two.

Take $C = A \mathbin{\&} B$ for the first theorem.

Take $C = \star$ for the second theorem.

Part III

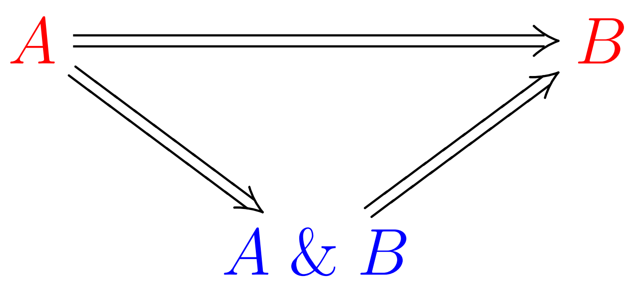
Threesomes, without blame

Threesomes

Let A, B, C, P, Q range over types.

$$v : A \stackrel{P}{\Longrightarrow} B \quad =_{\text{def}} \quad v : A \Rightarrow P \Rightarrow B$$

whenever $P <:_n A$ and $Q <:_n B$.



Combining threesomes

$$v : A \overset{P}{\Longrightarrow} B \overset{Q}{\Longrightarrow} C$$

$=$ Def'n: $P <:_n A, P <:_n B, Q <:_n B, Q <:_n C$

$$v : A \Rightarrow P \Rightarrow B \Rightarrow Q \Rightarrow C$$

$=$ 3'rd thm, $P \,\&\, Q <:_n B$

$$v : A \Rightarrow P \Rightarrow Q \Rightarrow C$$

$=$ 1'st thm

$$v : A \Rightarrow P \Rightarrow P \,\&\, Q \Rightarrow Q \Rightarrow C$$

$=$ 3'rd thm, $A \,\&\, P \,\&\, Q <:_n P, P \,\&\, Q \,\&\, C <:_n Q$

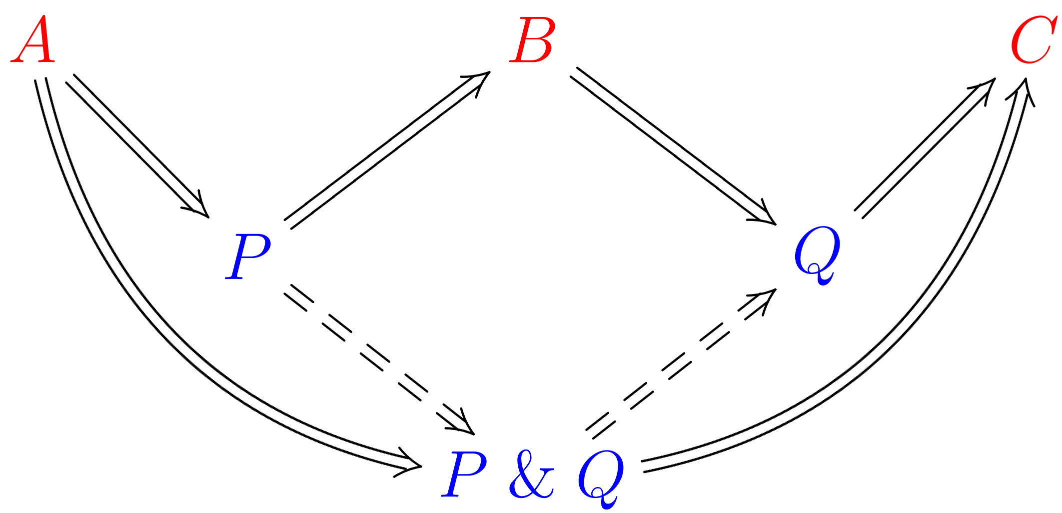
$$v : A \Rightarrow P \,\&\, Q \Rightarrow C$$

$=$ Def'n: $P \,\&\, Q <:_n A, P \,\&\, Q <:_n C$

$$v : A \overset{P\&Q}{\Longrightarrow} C$$

Combining threesomes

$$v : A \overset{P}{\Longrightarrow} B \overset{Q}{\Longrightarrow} C \quad = \quad v : A \overset{P \& Q}{\Longrightarrow} C$$



Recovering tail recursion!

$$\lceil even^\star\ 4 \rceil$$

$$\longrightarrow$$

$$(odd\ 3) : \texttt{Bool} \overset{\texttt{Bool}}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$\lceil even^\star\ 2 \rceil : \star \overset{\texttt{Bool}}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$(odd\ 1) : \texttt{Bool} \overset{\texttt{Bool}}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$\lceil even^\star\ 0 \rceil : \star \overset{\texttt{Bool}}{\Longrightarrow} \star$$

$$\longrightarrow$$

$$\lceil \texttt{true} \rceil$$

Syntax

Types	A, B, C, P, Q	$::=$	$\iota \mid A \to B \mid \star \mid \bot$
Terms	s, t	$::=$	$c \mid op(\vec{t}) \mid x \mid \lambda x{:}A.\, t \mid t\ s \mid$
			$\texttt{blame}\ p \mid s : A \stackrel{P}{\Longrightarrow} B$
Values	u	$::=$	$c \mid \lambda x{:}A.\, t$
	v, w	$::=$	$u \mid u : A \stackrel{P}{\Longrightarrow} \star \mid$
			$u : A \to B \stackrel{P \to Q}{\Longrightarrow} A' \to B'$

Reductions

$$
\begin{array}{rcl}
(u : A \to B \overset{P \to Q}{\Longrightarrow} A' \to B')\, w & \longrightarrow & u\,(w : A' \overset{P}{\Longrightarrow} A) : B \overset{Q}{\Longrightarrow} B' \\
u : A \overset{P}{\Longrightarrow} B \overset{Q}{\Longrightarrow} C & \longrightarrow & u : A \overset{P \& Q}{\Longrightarrow} C \\
u : \iota \overset{\iota}{\Longrightarrow} \iota & \longrightarrow & u \\
u : A \overset{\perp}{\Longrightarrow} B & \longrightarrow & \texttt{blame}
\end{array}
$$

Part IV

Threesomes, with blame

Adding blame

Types	A, B, C	$::=$	$\iota \mid A \to B \mid \star \mid \bot$
Labeled types	P, Q	$::=$	$\iota^p \mid P \to^p Q \mid \star \mid \bot^{pGq}$
Terms	s, t	$::=$	$c \mid op(\vec{t}) \mid x \mid \lambda x{:}A.\, t \mid t\ s \mid$
			$s : A \stackrel{P}{\Longrightarrow} B \mid s : G \mid \texttt{blame}\ p$

Key ideas

$$A \overset{\iota^p}{\Longrightarrow} B \quad \sim \quad A \overset{p}{\Rightarrow} \iota \Rightarrow B$$

$$A \overset{\perp^{Gpq}}{\Longrightarrow} B \quad \sim \quad A \overset{p}{\Rightarrow} G \overset{q}{\Rightarrow} \perp \Rightarrow B$$

Notation

$$
\begin{aligned}
||\iota^p|| &= \iota^p \\
||P \rightarrow^p G|| &= (\star \rightarrow \star)^p
\end{aligned}
$$

We write A^{G^p} when $||A|| = G^p$.

Meet with blame

$$
\begin{array}{rcll}
P \mathbin{\&\&} \star & = & P & \\
\star \mathbin{\&\&} P & = & P & \\
\iota^p \mathbin{\&\&} \iota^q & = & \iota^p & \\
P \to^p Q \mathbin{\&\&} P' \to^q Q' & = & (P' \mathbin{\&\&} P) \to^p (Q \mathbin{\&\&} Q') & \\
P^{Gp} \mathbin{\&\&} Q^{Hq} & = & \bot^{Gpq} & \text{if } G \neq H \\
\bot^{Gpq} \mathbin{\&\&} Q & = & \bot^{Gpq} & \\
P^{Gp} \mathbin{\&\&} \bot^{Gqr} & = & \bot^{Gpr} & \\
P^{Gp} \mathbin{\&\&} \bot^{Hqr} & = & \bot^{Gpq} & \text{if } G \neq H
\end{array}
$$

Reductions

$$(u : A \to B \overset{P \to^p Q}{\Longrightarrow} A' \to B')\, w \quad \longrightarrow \quad u\,(w : A' \overset{P}{\Longrightarrow} A) : B \overset{Q}{\Longrightarrow} B'$$

$$u : A \overset{P}{\Longrightarrow} B \overset{Q}{\Longrightarrow} C \quad \longrightarrow \quad u : A \overset{P \&\& Q}{\Longrightarrow} C$$

$$u : \iota \overset{\iota^p}{\Longrightarrow} \iota \quad \longrightarrow \quad u$$

$$u : A \overset{\bot^{Gpq}}{\Longrightarrow} B \quad \longrightarrow \quad \texttt{blame}\ q$$

An example

$$\mathtt{1} : \mathtt{Int} \stackrel{n}{\Rightarrow} \star \stackrel{o}{\Rightarrow} \mathtt{Bool} \stackrel{p}{\Rightarrow} \star \stackrel{q}{\Rightarrow} \mathtt{Int}$$

$$\longrightarrow \quad \mathtt{blame}\ o$$

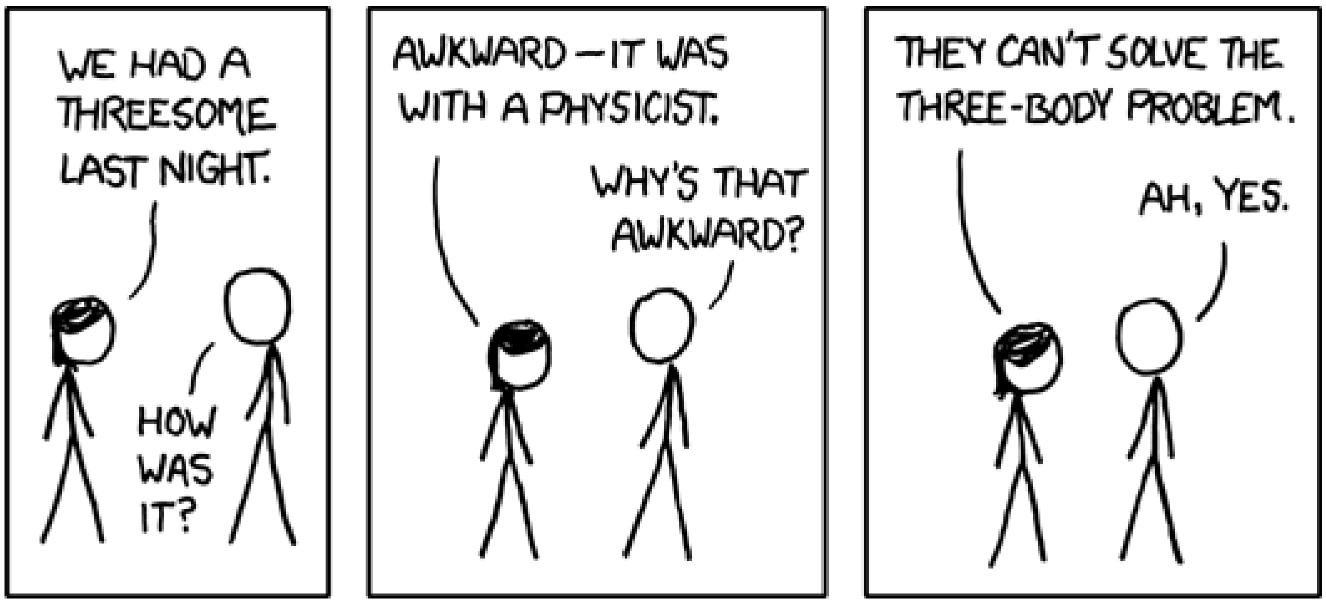
$$\mathtt{true} : \mathtt{Bool} \stackrel{n}{\Rightarrow} \star \stackrel{o}{\Rightarrow} \mathtt{Bool} \stackrel{p}{\Rightarrow} \star \stackrel{q}{\Rightarrow} \mathtt{Int}$$

$$\longrightarrow \quad \mathtt{blame}\ q$$

An example

$$
\begin{array}{ll}
 & 1 : \mathtt{Int} \overset{\mathtt{Int}^n}{\Longrightarrow} \star \overset{\mathtt{Bool}^o}{\Longrightarrow} \mathtt{Bool} \overset{\mathtt{Bool}^p}{\Longrightarrow} \star \overset{\mathtt{Int}^q}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & 1 : \mathtt{Int} \overset{\mathtt{Int}^n}{\Longrightarrow} \star \overset{\mathtt{Bool}^o}{\Longrightarrow} \mathtt{Bool} \overset{\perp^{\mathtt{Bool}\,pq}}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & 1 : \mathtt{Int} \overset{\mathtt{Int}^n}{\Longrightarrow} \star \overset{\perp^{\mathtt{Bool}\,oq}}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & 1 : \mathtt{Int} \overset{\perp^{\mathtt{Int}\,no}}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & \mathtt{blame}\ o
\end{array}
$$

$$
\begin{array}{ll}
 & \mathtt{true} : \mathtt{Bool} \overset{\mathtt{Int}^n}{\Longrightarrow} \star \overset{\mathtt{Bool}^o}{\Longrightarrow} \mathtt{Bool} \overset{\mathtt{Bool}^p}{\Longrightarrow} \star \overset{\mathtt{Int}^q}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & \mathtt{true} : \mathtt{Bool} \overset{\mathtt{Int}^n}{\Longrightarrow} \star \overset{\mathtt{Bool}^o}{\Longrightarrow} \mathtt{Bool} \overset{\perp^{\mathtt{Bool}\,pq}}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & \mathtt{true} : \mathtt{Bool} \overset{\mathtt{Bool}^n}{\Longrightarrow} \star \overset{\perp^{\mathtt{Bool}\,oq}}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & \mathtt{true} : \mathtt{Bool} \overset{\perp^{\mathtt{Bool}\,nq}}{\Longrightarrow} \mathtt{Int} \\
\longrightarrow & \mathtt{blame}\ q
\end{array}
$$


WE HAD A THREESOME LAST NIGHT.
HOW WAS IT?
AWKWARD — IT WAS WITH A PHYSICIST.
WHY'S THAT AWKWARD?
THEY CAN'T SOLVE THE THREE-BODY PROBLEM.
AH, YES.