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Graph Minors, Bidimensionality and Algorithms

PART IV

Warsaw, 2011

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Approximation algorithms

Definition: A c-**approximation** algorithm for a minimization problem is a polynomial-time algorithm that finds a solution of cost at most $\mathrm{OPT} \cdot c$.

Examples:

- $\frac{3}{2}$-approximation for Metric TSP,
- 2-approximation for Minimum Vertex Cover and Minimum Feedback Vertex Set
- $\frac{8}{7}$-approximation for Max 3SAT, etc.

- For some problems, we have lower bounds: there is no $(2-\epsilon)$-approximation for $\textsc{Vertex Cover}$ or $(\frac{8}{7}-\epsilon)$-approximation for $\textsc{Max 3SAT}$ (under suitable complexity assumptions).
- For some other problems, arbitrarily good approximation is possible in polynomial time: for any $c > 1$ (say, $c = 1.000001$), there is a polynomial-time c-approximation algorithm!

Approximation schemes

Definition: A **polynomial-time approximation scheme (PTAS)** for a problem P is an algorithm that takes an instance of P and a rational number $\epsilon > 0$,

- always finds a $(1+\epsilon)$-approximate solution,
- the running time is polynomial in n for every fixed $\epsilon > 0$.

Typical running times: $2^{1/\epsilon} \cdot n$, $n^{1/\epsilon}$, $(n/\epsilon)^2$, n^{1/ϵ^2}.

Some classical PTAS

- Vertex Cover for planar graphs
- TSP in the Euclidean plane
- Steiner Tree in planar graphs
- Knapsack

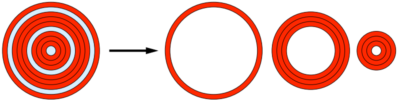
Shifting strategy

Classical approach: Baker [J. ACM 1994] and of Hochbaum and Maass [J. ACM 1985]

Example: Vertex Cover

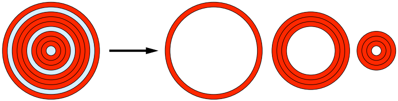
Fact: There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for $\textsc{Vertex Cover}$ for planar graphs.

Example: Vertex Cover



- Let $D := 1/(3\epsilon)$. For a fixed $0 \leq s < D$, delete every layer L_i with $i = s \pmod{D}$
- The resulting graph G_s is D-outerplanar, hence it has treewidth $O(D) = O(1/\epsilon)$.
- Using the $O(2^w \cdot n)$ time algorithm for $\textsc{Vertex Cover}$, the problem on G_s can be solved in time $2^{O(1/\epsilon)} \cdot n$.

Example: Vertex Cover



- For a fixed $0 \leq s < D$, define F_s as the graph induced by layers L_{i-1}, L_i, L_{i+1}, $i = s \pmod D$.
- The resulting graph is 3-outerplanar, hence it has treewidth $O(1)$.
- For at least one value of s, F_s contains at most $3/D = \epsilon$ vertices of some optimal solution.
- The union of vertex covers of F_s and G_s is a $(1+\epsilon)$-approximate solution.

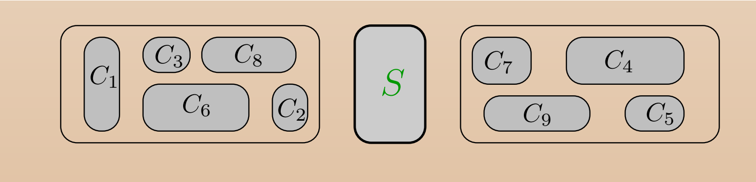
Let's take a different look

Bidimensionality and EPTAS

Branch-width separation

Lemma (Tree-width separation)

Let $G = (V, E)$ be a graph of treewidth t, and $w : V \to \{0, 1\}$ be a weight function. Then there is a set $S \subset V$ of size at most $t + 1$ such that the connected components $C_1, \ldots, C_\ell$ of $G[V \setminus S]$ can be grouped into two sets $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\frac{w(V)-w(S)}{3} \leq w(\mathcal{C}_i) \leq \frac{2(w(V)-w(S))}{3}$, for $i \in \{1, 2\}$.



Crucial Lemma

Lemma

Let X be a VC of size k in a planar graph G. For every $\varepsilon > 0$, $\exists$ set $X' \subseteq V(G)$ such that

- $|X'| \leq \varepsilon k$;
- $\mathbf{bw}(G \setminus X') = O(\frac{1}{\varepsilon})$

In other words, G has a constant-treewidth vertex removal set of size $O(k)$.

Put it a bit differently

$\forall \varepsilon > 0$, $\exists\ X' \subseteq V(G)$ s.t.

- $|X'| \le \varepsilon k$;
- $\mathbf{bw}(G \setminus X') = O(\frac{1}{\varepsilon})$

-
-

Put it a bit differently

$\forall \varepsilon > 0$, $\exists\ X' \subseteq V(G)$ s.t.

- $|X'| \leq \varepsilon k$;
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$\forall \varepsilon > 0\ \exists \delta > 0$, $X' \subseteq V(G)$ s.t.

- $|X'| \leq \varepsilon k$;
- $\forall$ component C of $G \setminus X'$, $|V(C) \cap X| \leq \frac{1}{\varepsilon^2}$

Indeed, X is bidimensional, and thus $\mathbf{bw}(C) = O\sqrt{|V(C) \cap X|} = O(\frac{1}{\varepsilon})$.

Proof

What we want to prove:

- $|X'| \leq \varepsilon k$;
- For every component C of $G \setminus X'$, $|V(C) \cap X| \leq \frac{1}{\varepsilon^2}$

If $k \leq \frac{1}{\varepsilon^2}$, we put $X' = \emptyset$.

Proof

Let $k > \frac{1}{\varepsilon^2}$.

Let $T(G, k)$ be the minimum size of the set X' s.t.

- For every component C of $G \setminus X'$, $|V(C) \cap X| = O(\frac{1}{\varepsilon^2})$

We prove that $T(G, k) \leq \varepsilon k - \delta\sqrt{k}$ for some $\delta > 0$.

Proof

We want to prove that $T(G,k) \le \varepsilon k - \delta\sqrt{k}$.

Let S be a balanced X-separator.

Then

$$T(G,k) \quad \le \quad |S| + \max_{1/3 \le \alpha \le 2/3} \left(T(G_1, \alpha k) + T(G_2, (1-\alpha)k)\right)$$

Proof

Take a weight function w assigning weight 1 to vertices of X and 0 to $V(G) \setminus X$.

By bidimensionality of VC, $\mathbf{bw}(G) = O(\sqrt{k})$. By Separation Lemma, G has a balanced X-separator of size at most $\beta\sqrt{k}$ for some $\beta > 0$.

Proof

Let S be a balanced X-separator of size at most $\beta\sqrt{k}$.

Then

$$\begin{aligned} T(G,k) &\leq |S| + \max_{1/3 \leq \alpha \leq 2/3} \left(T(G_1, \alpha k) + T(G_2, (1-\alpha)k)\right) \\ &\leq \beta\sqrt{k} + T(G_1, \frac{k}{3}) + T(G_2, \frac{2k}{3}) \end{aligned}$$

Proof

$$\begin{aligned}
T(G,k) &\leq |S| + \max_{1/3 \leq \alpha \leq 2/3} (T(G_1, \alpha k) + T(G_2, (1-\alpha)k)) \\
&\leq \beta\sqrt{k} + T(G_1, \frac{k}{3}) + T(G_2, \frac{2k}{3}) \\
&\leq \beta\sqrt{k} + (\varepsilon\frac{k}{3} - \delta\sqrt{\frac{k}{3}}) + (\varepsilon\frac{2k}{3} - \delta\sqrt{\frac{2k}{3}}) \\
&= \varepsilon k + \beta\sqrt{k} - \delta(\sqrt{\frac{k}{3}} - \sqrt{\frac{2k}{3}}) \\
&\leq \varepsilon k - \delta\sqrt{k}
\end{aligned}$$

for $\delta \geq \beta/(-1 + \sqrt{1/3} + \sqrt{2/3})$

Remark

If VC X is given, construction of to-constant-branchwidth-removal set X' can be done in polynomial time.

Algorithm

INPUT: Planar graph G, $\varepsilon > 0$

OUTPUT: vertex cover of size at most $(1+\varepsilon)OPT$

Use well-known 2-approximation to compute VC of G: X

Put $\varepsilon' = \varepsilon/2$ and use Lemma to compute set $X' \subseteq V(G)$ s.t.

- $|X'| \leq \varepsilon' k$;
- $\mathbf{bw}(G \setminus X') = O(\frac{1}{\varepsilon})$

Compute in time $O(2^{O(\frac{1}{\varepsilon})}n)$ optimum VC of $G \setminus X'$

Output $VC(G \setminus X') \cup X'$

Algorithm

$VC(G \setminus X') \cup X'$ is a VC in G of size

$$VC(G \setminus X') + |X'| \leq VC(G \setminus X') + \varepsilon'|X|$$

$$= VC(G) + \varepsilon'|X| \leq VC(G) + \varepsilon VC(G) = (1+\varepsilon)OPT$$

Usual questions

- What properties of Vertex Cover did we use?
- What properties of planar graph did we use?

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What properties of Vertex Cover did we use?

- $\mathbf{bw}(G) = O(\sqrt{k})$, or bidimensionality. But $\mathbf{bw}(G) = o(k)$ also will do
- Constant factor approximation
- "Separability", meaning that for separator S, and components $G[V \setminus S]$

$$\sum_{C} OPT(C) \leq OPT(G) + |S|$$

Again,

$$\sum_{C} OPT(C) \leq OPT(G) + \gamma|S|$$

will do

What properties of Vertex Cover did we use?

- On graphs of constant branchwidth the problem is solvable in polynomial time
- Remark: For EPTAS problem is FPT parameterized by branchwidth

What about FVS?

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$$\sum_C OPT(C) \leq OPT(G) + |S|.$$

OK

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- $\mathbf{bw}(G) = O(\sqrt{k})$, or bidimensionality. OK.
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Which means that FVS has PTAS on planar graphs!

What about Dominating Set?

- Should be a bit more careful to define separability property and use contraction bidimensionality

Crucial Lemma for DS

Lemma

Let X be a DS of size k in a planar graph G. For every $\varepsilon > 0$, $\exists$ set $X' \subseteq V(G)$ such that

- $|X'| \leq \varepsilon k$;
- $\mathbf{bw}(G \setminus X') = O(\frac{1}{\varepsilon})$

In other words, G has a constant-treewidth vertex removal set of size $O(k)$.

Proof

As for VC, we put $T(G,k)$ be the minimum size of set X'.

We want to prove that for some $\delta > 0$, $T(G,k) \leq \varepsilon k + \delta\sqrt{k}$.

Let S be a balanced X-separator. Instead of removal S, we contract!

Then

$$T(G,k) \quad \leq \quad |S| + \max_{1/3 \leq \alpha \leq 2/3} (T(G_1, \alpha k) + T(G_2, (1-\alpha)k))$$

Proof

Let S be a balanced X-separator.

By bidimensionality of DS, $\mathbf{bw}(G) = O(\sqrt{k})$. By separation lemma, G has a balanced X-separator of size at most $\beta\sqrt{k}$.

Proof

Let S be a balanced X-separator of size at most $\beta\sqrt{k}$.

Then

$$T(G,k) \quad \leq \quad \beta\sqrt{k} + T(G_1, \frac{k}{3}) + T(G_2, \frac{2k}{3})$$

Algorithm

INPUT: Planar graph G, $\varepsilon > 0$

OUTPUT: dominating set of size at most $(1+\varepsilon)OPT$

Use a (constant) c-approximation to compute DS of planar graph G: X

Put $\varepsilon' = \varepsilon / c$ and use Lemma to compute set $X' \subseteq V(G)$ s.t.

- $|X'| \leq \varepsilon' k$;
- $\mathbf{bw}(G \setminus X') = O(\frac{1}{\varepsilon})$

For each component C_i of $G \setminus X'$ define C_i' as contracting G on C_i.

Compute in time $O(2^{O(\frac{1}{\varepsilon})} n)$ optimum solution D of union of C_i'

Algorithm

$D \cup X'$ is a DS in G of size

$|D| + |X'| \leq |D| + \varepsilon' |X|$

$= DS(G) + \varepsilon' |X| \leq DS(G) + \varepsilon DS(G) = (1 + \varepsilon) OPT$

What about Connected Dominating Set?

Or shall we try to state a generic result?

Theorem

Let Π be a "reducible" minor- (contraction-) bidimensional problem with the "separation" property. There is an EPTAS for Π on planar graphs.

Where did we use planarity?

- Only for bidimensionality, i.e. the grid theorem

PTAS for Vertex Cover holds also on graphs excluding some fixed graph as a minor!

Where did we use planarity?

Theorem (FF, Lokshtanov, Raman, Saurabh, 2011)

Let Π *be a "reducible" minor- (contraction-) bidimensional problem with the separation property and* H *be a (apex) graph. There is an EPTAS for* Π *on the class of graphs excluding* H *as a minor.*

EPTAS on H-minor-free graphs

Feedback Vertex Set, Vertex Cover, Connected Vertex Cover, Cycle Packing, Diamond Hitting Set, Vertex-$\mathcal{H}$-Packing, Vertex-$\mathcal{H}$-Covering, Maximum Induced Forest, Maximum Induced Bipartite Subgraph, Maximum Induced Planar Subgraph ...

EPTAS on apex-minor-free graphs

Edge Dominating Set, Dominating Set, r-Dominating Set, q-Threshold Dominating Set, Connected Dominating Set, Directed Domination, r-Scattered Set, Minimum Maximal Matching, Independent Set, Maximum Full-Degree Spanning Tree, Max Induced at most d-Degree Subgraph, Max Internal Spanning Tree, Induced Matching, Triangle Packing ...

What we learned in this course?

- Graph Minors
- Implication of Graph Minors to Algorithmsto check in polynomial time properties closed under minors
- Branchwidth and its obstructions
- Grid theorem

What we learned in this course?

- Bidimensionality
- Use of bidimensionality to design subexponential parameterized algorithms
- Catalan structures and dynamic programming
- Bidimensionality and PTAS

Further reading. Bidimensionality and PTAS

- E. D. Demaine and M. Hajiaghayi, *Bidimensionality: new connections between FPT algorithms and PTASs*, SODA 2005, 590–601.
- F. V. Fomin, D. Lokshtanov, V. Raman, and S. Saurabh, *Bidimensionality and EPTAS*, SODA 2011.