FEDOR V. FOMIN

Graph Minors, Bidimensionality and Algorithms

PART III

Warsaw, 2011

Reminder

Back to the YEAR 2010...

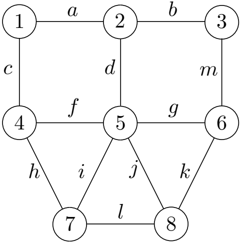
Branch Decompositions

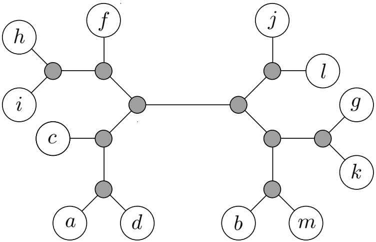
Definition

A branch decomposition of a graph $G = (V, E)$ is a tuple (T, μ) where

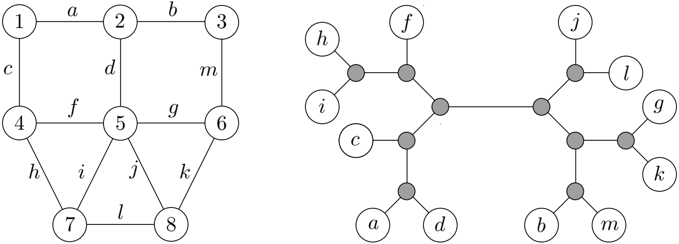
- T is a tree with degree 3 for all internal nodes.
- μ is a bijection between the leaves of T and $E(G)$.

Example of Branch Decomposition





Edge $e \in T$ partitions the edge set of G in A_e and B_e



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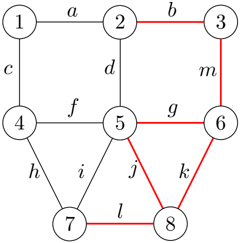
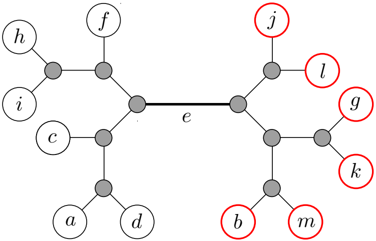
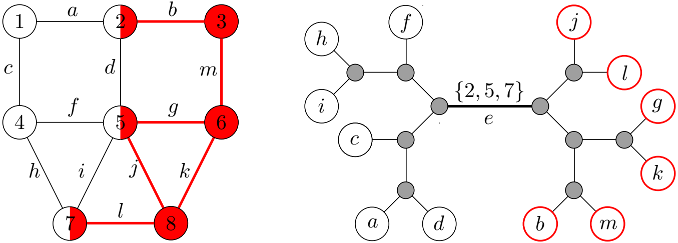




Middle set $\mathrm{mid}(e) = V(A_e) \cap V(B_e)$

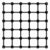


Branchwidth

- The *width* of a branch decomposition is $\max_{e \in T} |\operatorname{mid}(e)|$.
- The *branchwidth* of a graph G is the minimum width over all branch decompositions of G.

Grid Theorem

Theorem (Robertson, Seymour & Thomas, 1994)

Let $\ell \geq 1$ be an integer. Every planar graph of branchwidth $\geq 4\ell$ contains $_{\ell}$ *as a minor.*

Subexponential algorithms on planar graphs: What is the main idea?

Dynamic programming and Grid Theorem

Meta conditions

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

(B) For every graph $G \in \mathcal{G}$ and given a branch decomposition (T, μ) of G, the value of $P(G)$ can be computed in $f(\mathbf{bw}(T, \mu)) \cdot n^{O(1)}$ steps.

Algorithm

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

(B) For every graph $G \in \mathcal{G}$ and given a branch decomposition (T, μ) of G, the value of $P(G)$ can be computed in $f(\mathbf{bw}(T, \mu)) \cdot n^{O(1)}$ steps.

If $\mathbf{bw}(T, \mu) > \alpha \cdot \sqrt{k}$, then by (A) the answer is clear

Else, by (B), $P(G)$ can be computed in $f(\alpha \cdot \sqrt{k}) \cdot n^{O(1)}$ steps.

When $f(k) = 2^{O(k)}$, the running time is $2^{O(\sqrt{k})} \cdot n^{O(1)}$

Bidimensionality: The main idea

If the graph parameter is closed under taking minors or contractions, the only thing needed for the proof branchwidth/parameter bound is to understand how this parameter behaves on a (partially triangulated) grid.

Bidimensionality: Demaine, FF, Hajiaghayi, Thilikos, 2005

Definition

A parameter P is *minor bidimensional with density* δ if

1. P is closed under taking of minors, and
2. for the $(r \times r)$-grid R, $P(R) = (\delta r)^2 + o((\delta r)^2)$.

Bidimensionality: Demaine, FF, Hajiaghayi, Thilikos, 2005

Definition

A parameter P is called *contraction bidimensional with density* δ if

1. P is closed under contractions,
2. for any partially triangulated $(r \times r)$-grid R,

   $P(R) = (\delta_R r)^2 + o((\delta_R r)^2)$, and
3. δ is the smallest δ_R among all paritally triangulated

   $(r \times r)$-grids.

Bidimensionality

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

Lemma

If P is a bidimensional parameter with density δ then P satisfies property (A) for $\alpha = 4/\delta$, on planar graphs.

Proof.

Let R be an $(r \times r)$-grid.

$P(R) \geq (\delta_R r)^2$.

If G contains R as a minor, then $\mathbf{bw}(G) \leq 4r \leq 4/\delta\sqrt{P(G)}$. $\square$

Examples of bidimensional problems

Vertex cover

Dominating Set

Independent Set

(k, r)-center

Feedback Vertex Set

Minimum Maximal Matching

Planar Graph TSP

Longest Path ...

Bidimensional theory

If $\mathbf{P}$ is a parameter that

(A) is minor (contraction) bidimensional

(B) can be computed in $f(\mathbf{bw}(G)) \cdot n^{O(1)}$ steps.

then there is a $f(O(\sqrt{k})) \cdot n^{O(1)}$ step algorithm for checking

whether $\mathbf{P}(G) \leq k$ for H (apex) -minor free graphs.

We now fix our attention to property (B) and function f.

PART III

Dynamic programming and Catalan structures

Dynamic programming for branch decompositions

- We root the tree T of the branch decomposition (T, τ),
- We define a partial solution for each cut-set of an edge e of T
- We compute all partial solutions bottom-up (using the partial solutions corresponding to the children edges).

This can be done in $O(f(\ell) \cdot n)$ if we have a branch decomposition of width at most ℓ.

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This can be done in $O(f(\ell) \cdot n)$ if we have a branch decomposition of width at most ℓ.

$f(\ell)$ depends on the number of partial solutions we have to compute for each edge of T.

- To find a good bound for $f(\ell)$ is important!

For many problems, $2^{O(\mathbf{bw}(G))} \cdot n^{O(1)}$ step algorithms exist.

Dynamic programming on graphs with small branchwidth gives such algorithms for problems like

Vertex Cover,

Dominating Set, or

Edge Dominating Set, (and others...)

However: There are (many) problems where no general $2^{O(\mathbf{bw}(G))} \cdot n^{O(1)}$ step algorithm is known.

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Such problems are

Longest Path, Longest Cycle, Connected Dominating Set, Feedback Vertex Set, Hamiltonian Cycle, Max Leaf Tree and Graph Metric TSP

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Such problems are

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For the natural parameterizations of these problems, no $2^{O(\sqrt{k})} \cdot n^{O(1)}$ step FPT-algorithm follows by just using bidimensionality theory and dynamic programming.

Example: k- Path

The k-Path problem is to decide, given a graph G and a positive integer k, whether G contains a path of length k.

k- Path

On general graphs k-Path can be solved Color Coding technique in time $O((2e)^k n)$ [Alon, Yuster, Zwick, 1995].

A chain of improvements up to recent $O^*(1.66^k)$ algorithm of [Björklund, Husfeldt, Kaski, Koivisto, 2010]

k-Path

On planar graphs k-Path has a

$2^{O(\sqrt{k} \cdot \log k)} \cdot n^{O(1)}$ step algorithm.

Because

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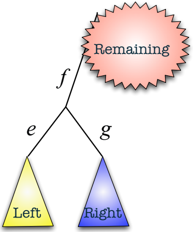
Because

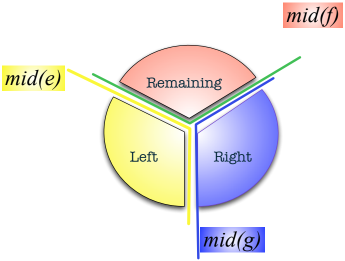
(A) The parameter is minor bidimensional

(B) to find a longest path in a graph G takes

$2^{O(\mathbf{bw}(G) \cdot \log\ \mathbf{bw}(G))} \cdot n$ steps

Dynamic programming: k-Path





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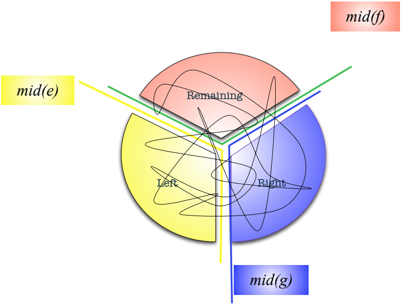


k-Path

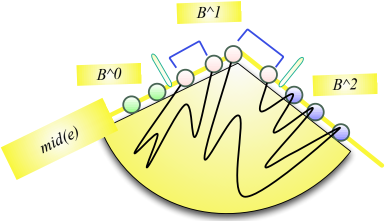
Fact: Given a branch decomposition of width w, k-PATH can be solved in time $w^{O(w)} \cdot n$.

G_e: subgraph formed by edges of the subtree rooted at e.

If P is a path, then the subgraph

$P_e = G_e \cap P$ is a set of paths with

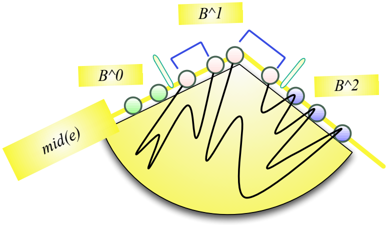
endpoints in $\mathrm{mid}(e)$.



k-Path

What are the important properties of P_e "seen from the outside world"?

- The subsets B_e^0, B_e^1, B_e^2 of $\mathrm{mid}(e)$ having degree 0, 1, and 2.
- Disjoint pairs $\texttt{pairs}(e)$ of B_e^1.



Number of subproblems $(B_e^0, B_e^1, B_e^2, \texttt{pairs}(e))$ for each edge e: at most $3^w \cdot w^w$.

The running time of dynamic programming is proportional to the amount of all possible subproblems $(B_e^0, B_e^1, B_e^2, \texttt{pairs}(e))$, which is $3^w \cdot w^w = 2^{O(w \log w)}$.

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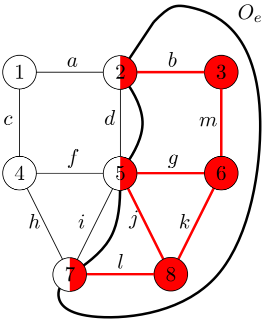
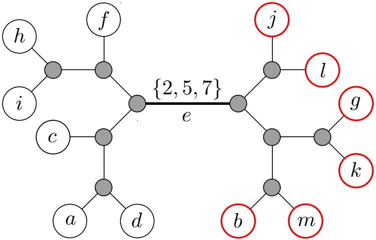
- Issue: The same problem appears in many dynamic programming algorithms
- Idea: as long as we care about sparse graph classes, we can take their structure into consideration
- We want to show that on planar graphs, the amount of pairs $\texttt{pairs}(e)$ in $\mathrm{mid}(e)$ is $2^{O(w)}$

Sphere-cut decomposition

Let G be a planar graph embedded on the sphere (or a plane) $\mathcal{S}_0$

A *sphere-cut decomposition* of G is a branch decomposition (T, τ) where for every $e \in E(T)$, the vertices in $\mathrm{mid}(e)$ are the vertices in a Jordan curve of $\mathcal{S}_0$ that meets no edges of G (a noose).





Seymour-Thomas 1994, Dorn-Penninkx-Bodlaender-FF 2005

Theorem

Every planar graph G of branchwidth ℓ has a sphere-cut decomposition of width ℓ. This decomposition can be constructed in $O(n^3)$ steps.

Proof ideas

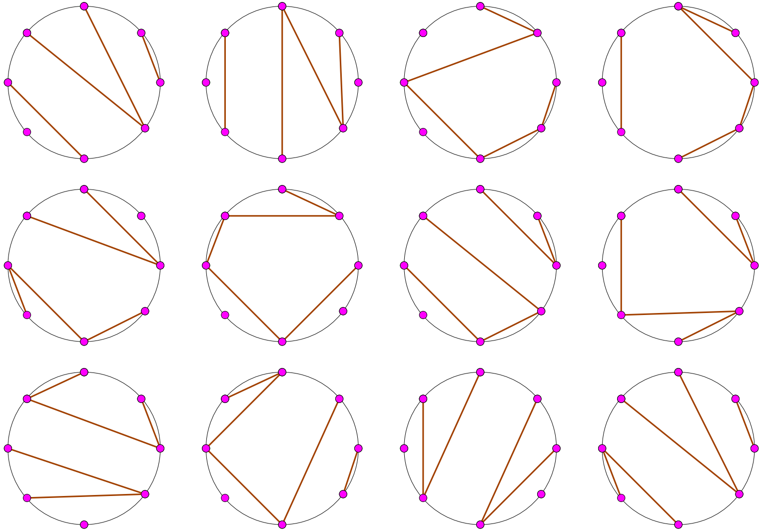
- Carving-width
- Connectivity properties of carving-width decompositions
- Branch-width as the carving width of a radial graph

Bounding $(B_e^0, B_e^1, B_e^2, \texttt{pairs}(e))$

We now have that

1: the vertices of $\mathrm{mid}(e)$ lay on the boundary of a disk and

2: the pairings $\texttt{pairs}(e))$ cannot be crossing because of planarity.



It follows that $\texttt{pairs}(e) = O(C(|\operatorname{mid}(e)|)) = O(C(w))$

Where $C(w)$ is the w-th *Catalan Number*.

It is known that $C(w) \sim \frac{4^w}{w^{3/2}\sqrt{\pi}} = 2^{O(w)}$

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We conclude that for planar graphs, there is an optimal branch decomposition of width w such that for every edge e, the amount of tuples $(B_e^0, B_e^1, B_e^2, \texttt{pairs}(e))$ is $2^{O(w)}$

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Therefore: dynamic programming for finding a k-path of a planar graph G on a sphere cut decompositions of G with width $\leq w$ takes $O(2^{O(w)} \cdot n)$ steps.

Conclusion:

Planar k-Path can be solved in $O(2^{O(\sqrt{k})} \cdot nn^{O(1)})$ steps

Algorithm:

- Compute branch-width of G in polynomial time. If $\mathbf{bw}(G) \geq 4\sqrt{k}$, then G contains $\sqrt{k} \times \sqrt{k}$ gris as a minor. Thus it also contains a path of length k
- If $\mathbf{bw}(G) \leq 4\sqrt{k}$, use sphere-cut decomposition to find in time $2^{O(\sqrt{k})}n$ if G has a path of length k.

- Similar results hold for several other problems where an analogue of $\texttt{pairs}(e)$ can be defined for controlling the size of the tables in dynamic programming.

- Similar results hold for several other problems where an analogue of `pairs`(e) can be defined for controlling the size of the tables in dynamic programming.

- Like that one can design $2^{O(\sqrt{k})} \cdot n^{O(1)}$ step algorithms for parameterized planar versions of Cycle Cover, Path Cover, Longest Cycle, Connected Dominating Set, Feedback Vertex Set, Graph Metric TSP, Max Leaf Tree, etc.

Planarity not a limit

- For H-minor free graphs, one can construct an algorithm that solves the k-$\textsc{Path}$ problem in $2^{O(\sqrt{k})} \cdot n^{O(1)}$ steps.

More grids

Grids for other problems or walls

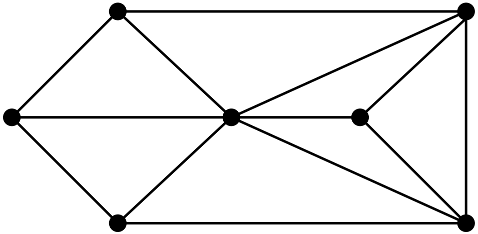
t-spanners

Definition (t-spanner)

Let t be a positive integer. A subgraph S of G, such that $V(S) = V(G)$, is called a t-spanner, if $\mathrm{dist}_S(u,v) \leq t \cdot \mathrm{dist}_G(u,v)$ for every pair of vertices u and v. The parameter t is called the *stretch factor* of S.

Examples of spanners

3 and 2-spanners



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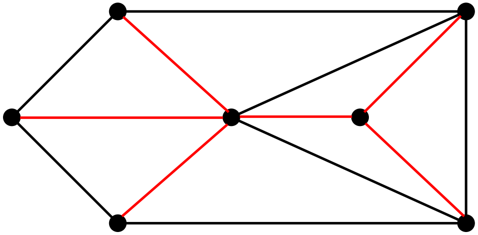

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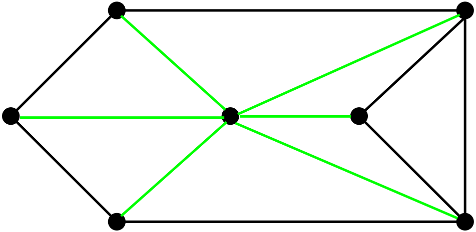


Spanners of bounded branchwidth

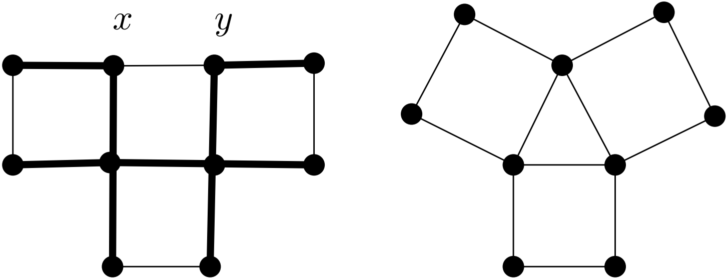
Problem (k-Branchwidth t-spanner)

Instance: A connected graph G and positive integers k and t.

Question: Is there a t-spanner of G of branchwidth at most k?

Topological minors

Contracting edge can increase t

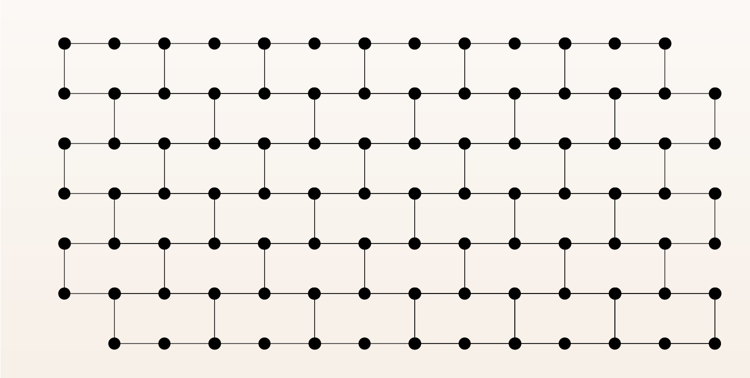


Planar graphs

Theorem (Bounds for planar graphs)

Let G be a planar graph of branchwidth k and let S be a t-spanner of G. Then the branchwidth of S is $\Omega(k/t)$.

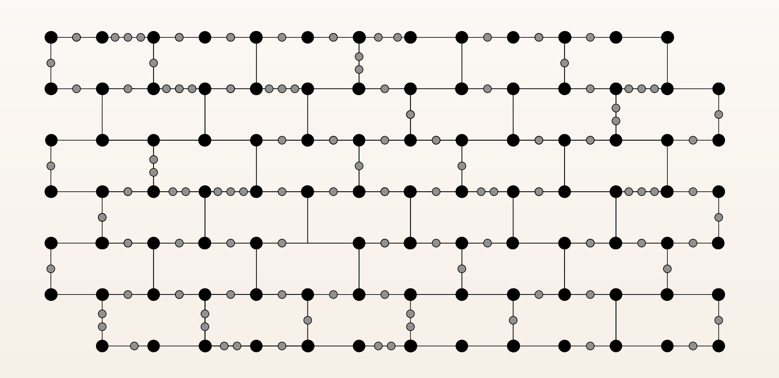
Walls



A wall of height k.

Walls

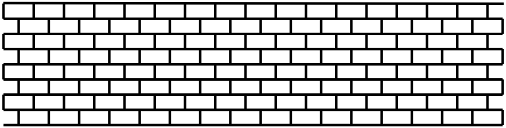
Observation: If a graph contains a $k \times k$ grid as a minor, it contains a wall of height k as a topological subgraph.



Which can be seen as a subgraph that is a subdivided wall.

Sketch of the proof

Walls and grids



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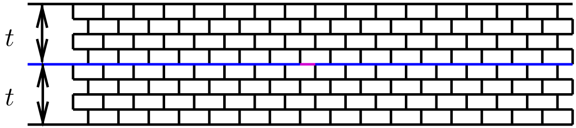

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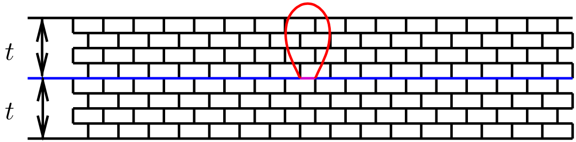

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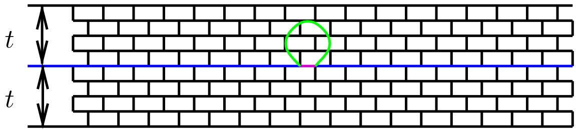

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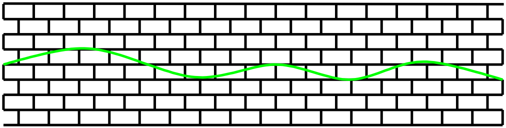

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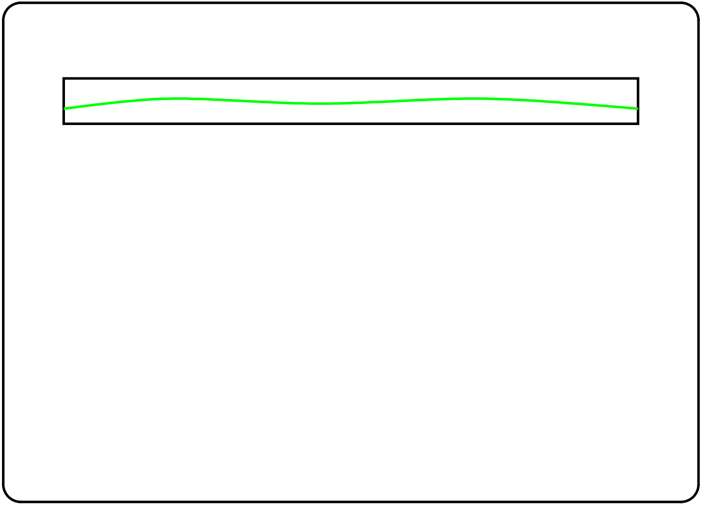

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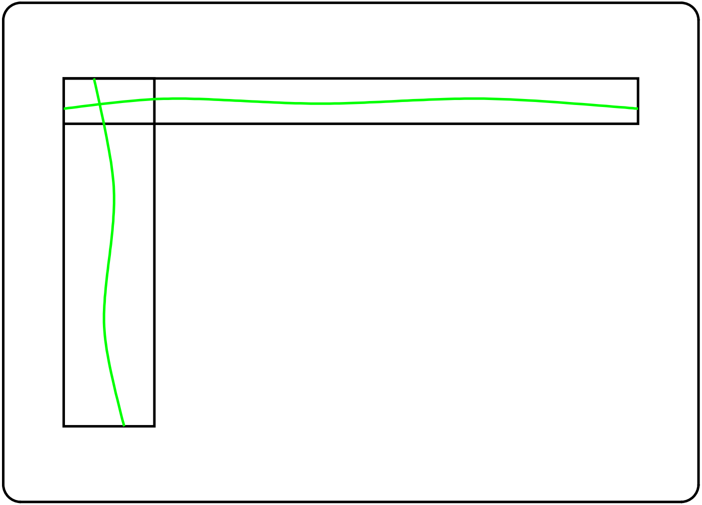

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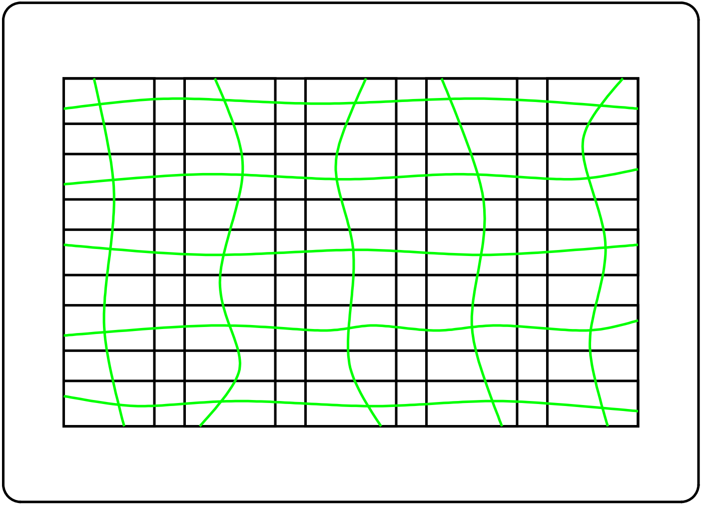

Algorithmic consequences

Theorem (Dragan, FF, Golovach, 2008)

Deciding if a planar graph G *has a* t*-spanner of treewidth at most* k *is solvable in time* $O(f(k,t) \cdot n^{O(1)})$.

Disjoint Path Problem

p-Disjoint Paths Problem

Instance: a graph G and terminals $(s_i, t_i) \in V(G)^2, i = 1, \ldots, k$.

Parameter: k

Question: Does G have k disjoint paths between s_i and t_i, $i = 1, \ldots, k$?

Disjoint Path Problem

- To solve problem on planar graphs: If the branch-width is small, do DP

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- If the branch-widht is large, there is a wall of large height
- If there is a wall of large height, there is a wall of large height containing no terminal vertices

Disjoint Path Problem

- To solve problem on planar graphs: If the branch-width is small, do DP
- If the branch-widht is large, there is a wall of large height
- If there is a wall of large height, there is a wall of large height containing no terminal vertices
- A vertex in the center of this wall is irrelevant. i.e. its removal does not change the problem

Conclusion

- WIN/WIN approach via Excluding Grid Theorem
- Dynamic programming and Catalan structures
- Working on planar problems don't afraid to hit the wall!

Further reading. Subexponential algorithms and bidimensionality

- J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks, and R. Niedermeier, *Fixed parameter algorithms for dominating set and related problems on planar graphs*, Algorithmica, 33 (2002), pp. 461–493.
- E. D. Demaine, F. V. Fomin, M. Hajiaghayi, and D. M. Thilikos, *Subexponential parameterized algorithms on graphs of bounded genus and H-minor-free graphs*, Journal of the ACM, 52 (2005), pp. 866–893.

Further reading. Catalan structures and dynamic programming

- F. Dorn, E. Penninkx, H. Bodlaender, and F. V. Fomin, *Efficient exact algorithms on planar graphs: Exploiting sphere cut branch decompositions*, Proceedings of the 13th Annual European Symposium on Algorithms (ESA 2005), vol. 3669 of LNCS, Springer, 2005, pp. 95–106.
- F. Dorn, F. V. Fomin, and D. M. Thilikos, *Catalan structures and dynamic programming on H-minor-free graphs*, in Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2008), ACM-SIAM, pp. 631–640.

Further reading. Surveys

- E. Demaine and M. Hajiaghayi, *The bidimensionality theory and its algorithmic applications*, The Computer Journal, (2007), pp. 332–337.
- F. Dorn, F. V. Fomin, and D. M. Thilikos, *Subexponential parameterized algorithms*, Computer Science Review, 2 (2008), pp. 29–39.