FEDOR V. FOMIN

Graph Minors, Bidimensionality and Algorithms

PART I

Warsaw, 2010

Outline of the lectures

- Part I: Graph Minors

  WQO

  Kraskul's theorem

  Robertson-Seymour Graph Minors Theorem

  Obstructions

  Meta-algorithmic consequences

Outline of the lectures

- Part II: Bidimensionality

  Branch-width

  Dynamic programming

  Excluding Planar grid

  Parameterized Algorithms

Outline of the lectures

- Part III: Bidimensionality

  Sphere-cut decompositions and Catalan structure

  EPTAS

  Kernelization (if time allows)

PART I

Wagner's Conjecture and Graph Minors series

W.Q.O. definition

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- $(x_i x_j)$: a good pair of x . Sequence x is a good sequence.

Proposition

A quasi-ordering X is a well-quasi-ordering if and only if X contains

- neither infinite antichain
- nor strictly decreasing sequence $x_0 > x_1 > \dots$.

PROOF: Ramsey arguments

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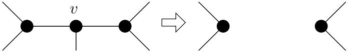
1. $\backslash v$: vertex removal: 

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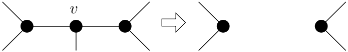
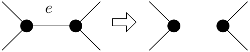
1. $\backslash v$: vertex removal: 

2. $\backslash e$: edge removal: 

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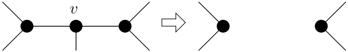
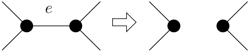
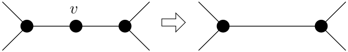
1. $\backslash v$: vertex removal: 

2. $\backslash e$: edge removal: 

3. $/v$: deg-2 vertex dissolution: 

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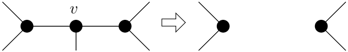

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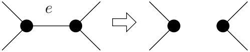

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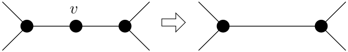

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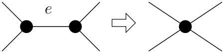

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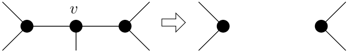
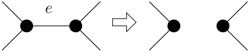
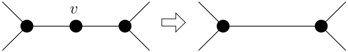
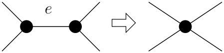
1. $\backslash v$: vertex removal: 

2. $\backslash e$: edge removal: 

3. $/v$: deg-2 vertex dissolution: 

4. $/e$: edge contraction 

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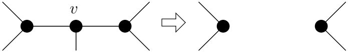
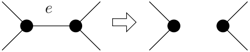
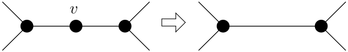
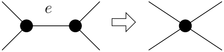
1. $\backslash v$: vertex removal: 

2. $\backslash e$: edge removal: 

3. $/v$: deg-2 vertex dissolution: 

4. $/e$: edge contraction 

Let $L \subseteq \{\backslash v, \backslash e, /v, /e\}$

$H \leq_{\mathcal{L}} G$ if H can be obtained from G after a sequence of operations in $\mathcal{L}$

Relation	Notation	$\backslash v$	$\backslash e$	$/v$	$/e$	WQO
induced subgraph	$(H \subseteq_{in} G)$	•				NO
subgraph	$(H \subseteq_{sb} G)$	•	•			NO
spanning subgraph	$(H \subseteq_{sp} G)$		•			NO
induced topological minor	$(H \leq_{it} G)$	•		•		NO
topological minor	$(H \leq_{tp} G)$	•	•	•		NO
induced minor	$(H \leq_{in} G)$	•			•	NO
contraction	$(H \leq_{cn} G)$				•	NO
minor	$(H \leq_{mn} G)$	•	•		•	YES

The fact that graphs are WQO by the minor relation was known as *The Wagner's Conjecture* formulated by Klaus Wagner in the 1930s (?).

The Graph Minors Series

The conjecture was proven by *Neil Robertson* and *Paul Seymour* in their Graph Minor series of papers.


Now it is known as the *Robertson & Seymour Theorem*.

Width of the proof: < 10 cm (23 papers)

Kruskal's theorem

Theorem (Kruskal, 1960)

Trees are W.Q.O. by the topological minor relation.

(Formerly also known as the Vázsonyi conjecture)

Proof of Kruskal's theorem

We prove the following stronger statement:

- Rooted trees are W.Q.O. by the topological minor containment

Trees T and T' with roots r and r'.

$T \leq T'$ if $\exists$ isomorphism φ from some subdivision of T to a subtree of T' preserving the tree-order on $V(T)$ associated with r and T.

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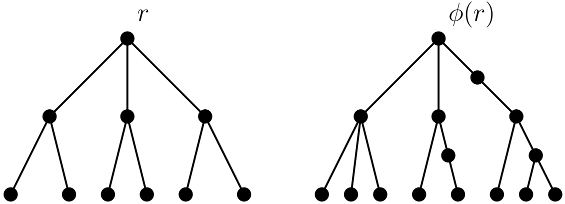

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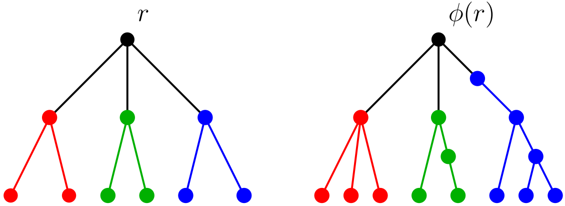

Proof sketch:

Suppose that there is a bad sequence of rooted trees. For $n \geq 0$ we select inductively the following bad sequence:

- Assume we constructed a sequence $T_0, T_1, T_2, \ldots, T_{n-1}$ s.t. there is a bad sequence starting with $T_0, T_1, T_2, \ldots, T_{n-1}$

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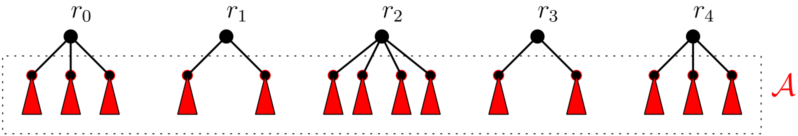
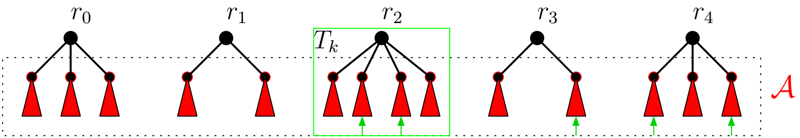
- $(T_n)_{n \geq 0}$ is a bad sequence

- For $(T_n)_{n\geq 0} = T_0, T_1, T_2, \ldots$
- A_n the set of components $T_n - r_n$.
- $\mathcal{A} = \bigcup_{n\geq 0} A_n$



- We first prove that $\mathcal{A}$ is WQO.

Proof that $\mathcal{A}$ is WQO



Let $(T^k)_{k \geq 0} = T^1, T^2, \ldots$ be a sequence of rooted trees in $\mathcal{A}$

For k define $n(k)$ to be the minimum n such that $T^k \in A_n$.

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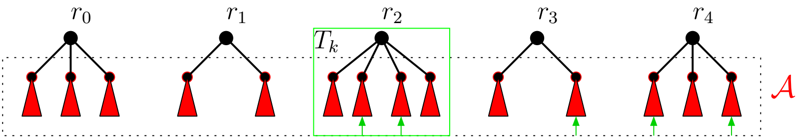

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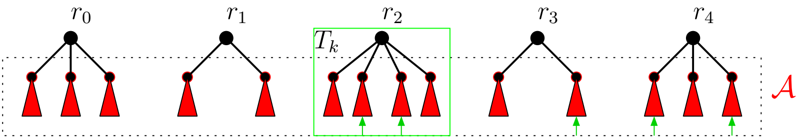

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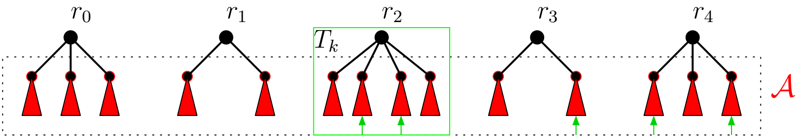
Choose k with the smallest $n(k)$.

Then the sequence $T_0, \ldots, T_{n(k)-1}, T^k, T^{k+1}, \ldots$ is good (by minimality of $T_{n(k)}$ and because $T^k \subset T_{n(k)}$).

Proof that $\mathcal{A}$ is WQO



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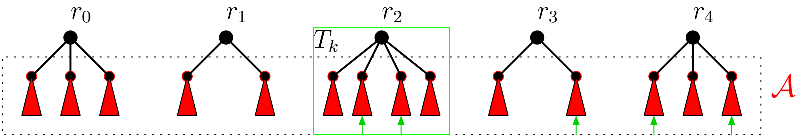
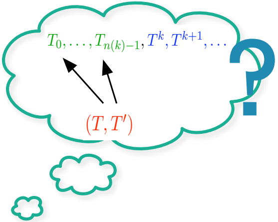


The sequence $T_0, \ldots, T_{n(k)-1}, T^k, T^{k+1}, \ldots$ is good

Thus it has a good pair (T, T')

Proof that $\mathcal{A}$ is WQO: $T \not\leq T_0, \ldots, T_{n(k)-1}$



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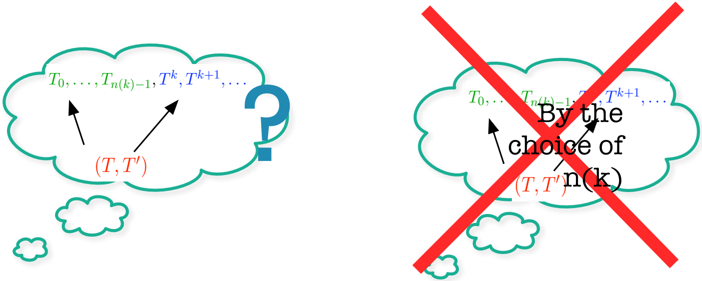


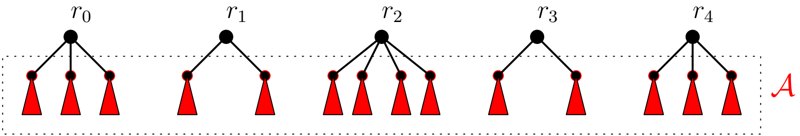
If $T_j = T' = T = T^i \leq T_{n(i)}$, $j < k \leq i$, then because $n(i) \geq n(k)$, we have $T_j \leq T_{n(i)}$.

Proof that $\mathcal{A}$ is WQO:

$$T_0, \ldots, T_{n(k)-1}, T^k, T^{k+1}, \ldots$$

- Thus both T' and T are from $T^k, T^{k+1}, \ldots$.
- Hence, $\mathcal{A}$ is WQO.

Proof of Kruskal Theorem, contd.:



- We know that $\mathcal{A}$ is WQO.
- next step, show that $[\mathcal{A}]^{<\omega}$, the set of all finite subsets of $\mathcal{A}$, is WQO.

Proof of Kruskal Theorem, contd.:

- $[\mathcal{A}]^{<\omega}$, the set of all finite subsets of $\mathcal{A}$, is WQO.
- For sets $A, B \in \mathcal{A}$, $A \leq B$ if there is an injective mapping $f\colon A \to B$ s.t. $a \leq f(a)$, $\forall a \in A$

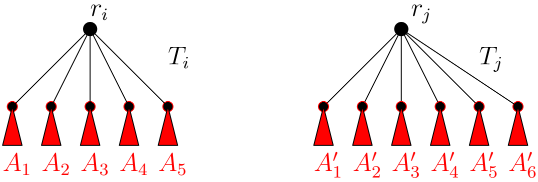
Lemma

If $\mathcal{A}$, is WQO then so is $[\mathcal{A}]^{<\omega}$

Proof of Kruskal Theorem, contd.:

Suppose (for a minute) that Lemma "$\mathcal{A}$ is WQO then so is $[\mathcal{A}]^{<\omega}$" holds. Then $[\mathcal{A}]^{<\omega}$ is WQO.

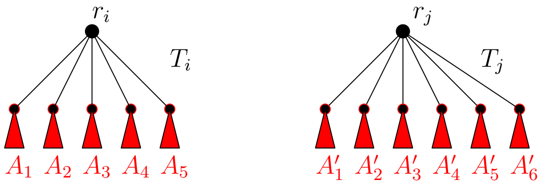
- The sequence $(A_n)_{n\geq 0}$ in $[\mathcal{A}]^{<\omega}$ should have a good pair (A_i, A_j): $A_i \leq A_j$, $i < j$
- $\exists$ injective mapping $f\colon A_i \to A_j$ s.t. $T^x \leq f(T^x)$, $\forall T^x \in A_i$



Proof of Kruskal Theorem, contd.:

Extend f to φ by $\varphi(r_i) = \varphi(r_j)$

- $T_i \leq T_j$
- (T_i, T_j) is a good pair in a bad sequence - CONTRADICTION!



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PROOF: Assume that $\mathcal{A}$ is WQO but $[\mathcal{A}]^{<\omega}$ not

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For every $i \geq 1$ choose A_i s.t. $A_0, A_1, \ldots, A_i$ is a bad sequence

and $|A_i| \to \min$

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pick $a_n \in A_n$, and set $B_n = A_n \setminus a_n$.

$(a_n)_{n \geq 0}$ is a good sequence, pick up an infinite increasing

subsequence $(a_{n_i})_{i \geq 0}$

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PROOF: take

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Good pair cannot be of the form (A_i, A_j) and of the form

$(A_i, B_j \leq A_j)$

But good pair (B_i, B_j) and $a_i \leq a_j$ imply that

$(A_i = B_i, \cup a_i, A_j = B_j \cup a_j)$ is a good pair.

REMARKS

The proof is non-constructive.

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Friedman (2002) observed that Kruskal's theorem has special cases that can be stated but not proved in first-order arithmetic.

(Though they can easily be proved in second-order arithmetic.)

How can we go further than trees?

We have to define the tree-likeness of a graph.

Treewidth of a graph

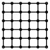
Theorem (Robertson & Seymour, GM IV)

For every $k \geq 0$, graphs of treewidth at most k are WQO.

What if the treewidth is unbounded?

Start from planar graphs.

Theorem (Robertson & Seymour, GM IV)

There is function f such that for every $k \geq 0$, a planar graph G of treewidth at least $f(k)$ contains $_k$ *as a minor.*

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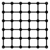
There is function f such that for every $k \geq 0$, a planar graph G of treewidth at least $f(k)$ contains $_k$ *as a minor.*

We will be back to this theorem soon...

What if the treewidth is unbounded?

Observation: *Every planar graph G on n vertices is a minor of sufficiently large grid.*

Wagner's conjecture for planar graphs, GM IV

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- Let $G_0, G_1, \ldots,$ be a bad sequence of planar graphs.
- If G_i, $i \geq 1$, is of treewidth more than $f(G_0)$, it has large grid as a minor, which contains G_0 as a minor. Thus $G_0 \leq G_i$.

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- If G_i, $i \geq 1$, is of treewidth more than $f(G_0)$, it has large grid

as a minor, which contains G_0 as a minor. Thus $G_0 \leq G_i$.

- All graphs in $G_1, G_2 \dots$ should have

treewidth at most $c = f(V(G_0))$, and thus are WQO!

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Answer: YES and NO!

NO: We should now also deal with graphs embedded in surfaces!

YES: But surfaces that are arranged together as trees!

Further reading

- Chapter 12 of Graph Theory, R. Diestel, 3rd edition.

Algorithmic consequences of the R&S theorem

A *graph parameter* $\mathbf{p}$ is a function mapping graphs to positive integers.

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A graph parameter is *minor-closed* if $H \leq_{mn} G \Rightarrow \mathbf{p}(H) \leq \mathbf{p}(G)$

Each parameter $\mathbf{p}$ corresponds to a parameterized problem:

p-PROBLEM OF DECIDING **p**

Instance: a graph G and an integer $k \geq 0$.

Parameter: k

Question: **p**(G)$\leq k$?

We say that a parameterized (by k) problem is FPT (fixed parameter tractable) if it can be solved in time

$$O(f(k) \cdot n^{O(1)}) \text{ steps}$$

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- Not all parameterized problems admit FPT-algorithms.

There are parameterized complexity classes like W[1], W[2], or W[P] and adequate reductions such that when a problem is hard for them is not expected to have an FPT-algorithm.

Minor-closed parameters:

- vertex cover, **vc**(G)
- feedback vertex set, **fvs**(G)
- branchwidth, **bw**(G)
- minimum maximal matching, **mmm**(G)
- k-almost$_{\Pi}(G) = \min\{|S| \mid G - S \in \Pi\}$ (Π is any minor-closed class)
- the genus of a graph, $\gamma(G)$
- $\mathbf{p}(G) = \min\{k \mid P_k \not\leq_{mn} G\}$

- Consequence of R&S Theorem: for any minor-closed graph class $\mathcal{G}$ the set of graphs not in it have a finite set of minor-minimal elements.
- we denote this set $\mathbf{ob}(\mathcal{G})$ and we call it obstruction set of $\mathcal{G}$.
- Observe: $\mathcal{G}$ if is minor-closed then $\mathbf{ob}(\mathcal{G})$ is finite.

Examples of obstruction sets

- Trees: K_3
- Outerplanar Graphs: $K_{2,3}$, K_4
- $\mathbf{bw}(G) \leq 2$: K_4
- planar graphs: $K_{3,3}$, K_5 (Theorem Kuratowski-Понтрягин)
- link-free graphs: 7 graphs (Petersen family: X-Y transformations of K_6)

Examples of obstruction sets

- Graphs with a vertex cover of size ≤ 5: 56 graphs
- Graphs with a vertex cover of size ≤ 6: 260 graphs
- Graphs with a feedback vertex of size ≤ 3: $\geq$ 744 graphs
- Graphs embeddable in the projective plane: 35 graphs
- Graphs embeddable in the torus: ≥ 2200 graphs

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- Graphs with branchwidth $\leq k$ graphs: obstructions have size $\leq (6^k - 1)/5$.

[Geelen, Gerards, Robertson, and Whittlee, JCTSB' 03]

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- Graphs with a vertex cover $\leq k$ graphs: obstructions have size $\leq 2(k+1)$.

[Michael J. Dinneen, Rongwei Lai, Disc. Math, '07]

Lower Bounds?

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- Searching an active and visible (or pathwidth) $\leq k$: $\geq (k!)^2$ graphs.

[Parsons '98], [Ellis, Sudborough, &Turner '94], [Takahashi, Ueno, & Kajitani, 94]

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- Searching an active and visible (or treewidth) $\leq k$: $2^{\Omega(k \cdot \log k)}$ graphs.

[Arvind Gupta, Damon Kaller, and Thomas Shermer, ICALP'99]

Main meta-algorithmic consequence of GM

Theorem

If $\mathbf{p}$ *is a minor-closed parameter, then* p-Problem of Deciding $\mathbf{p}$ *is in* FPT *by an* $O(f(k) \cdot n^3)$ *step algorithm.*

Moreover, for planar inputs (and more) the above algorithms are linear.

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- Let $\mathbf{ob}(\mathcal{G}_k)$ has at most $f(k)$ elements of size at most $g(k)$.
- $G \in \mathcal{G}_k$ iff $\forall_{H \in \mathbf{ob}(\mathcal{G}_k)}\ H \not\leq_{mn} G$

An algorithm for p-PROBLEM OF DECIDING **p**.

Decide-**p**(G, k)

1. for all $H \in \mathbf{ob}(\mathcal{G}_k)$

2. check (in $O(h(k) \cdot n^3)$ steps) whether $H \leq_{mn} G$

3. if the answer if YES, then output NO

4. output YES.

Applications

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Instance: a graph G and a positive integer k.

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Drawbacks of graph minors meta-algorithm:

1. the above proof is non-constructive as we do not know $\mathbf{obs}(\mathcal{G}_{k)}$
2. we know $\mathbf{obs}(\mathcal{G}_{k)}$ for few classes and for small values of k
3. when we have estimations of $f(k)$ and $g(k)$, they are immense. The corresponding FPT-algorithms have a heavy parameter dependance.

Conclusion:

However spectacular such unexpected solutions to long-standing problems may be, viewing the graph minor theory merely in terms of its corollaries will not do its justice. At least as important are the techniques developed for its proof...

Reinhard Diestel, Graph Theory, 4th edition.

Conclusion:

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Otherwise, exploit the structure of the obstruction to the small treewidth!

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Seems that the consequences of GM are great but completely unpractical.

But it appeared that the WIN/WIN approach of GM: either small treewidth or big obstruction is worth to try!

Further reading. Kruskal's Theorem and Graph Minors

R. Diestel, *Graph Theory*, Third Edition, Springer, Chapter 12

Further reading. Algorithmic Applications of Graph Minors

R. Downey and M. Fellows, *Parameterized Complexity*, Springer