1. Linear consistency testing.

   A triple of functions $f, g, h : \mathbb{Z}_2^n \to \mathbb{Z}_2$ are said to be *linear-consistent* if for every $x, y \in \{0,1\}^n$ it is the case that $f(x) + g(y) = h(x+y)$.

   a) Give a characterization of linear-consistent triples (f, g, h). Specifically, give upper bounds on the degree of f, g and h as polyomials over $\mathbb{Z}_2^n$. Give relationships that should hold between their coefficients.

   b) The linear consistency test is the following: Pick $x, y \in \mathbb{Z}_2^n$ at random and reject if $f(x) + g(y) \neq h(x+y)$. Give a upper bound on the acceptance probability of this test in terms of the Fourier coefficients of the first function f.

2. Here we give a simple, but weaker, test for a function being a dictator.

   a) Prove that a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is a dictator if and only if the following conditions hold: (i) For every $x, y \in \mathbb{Z}_2^n$, it is the case that $f(x)+f(y) = f(x+y)$, and (ii) For every $x, y \in \mathbb{Z}_2^n$, it holds that $f(x)f(y) = f(x \cdot y)$ where $x \cdot y = \langle x_1 y_1 \ldots, x_n y_n \rangle$.

   b) Analyze the following test for dictatorship: Given oracle for a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$, pick random $x, y, z \in \mathbb{Z}_2^n$ and accept if both the following conditions hold: (i) $f(x)+f(y) = f(x+y)$ and (ii) $f(x)f(y) = f(x \cdot y + z) - f(z)$.

3. Recycling queries in linearity test: Our goal is to query the function f on at most 6 values, accept linear functions with probability 1, but reject functions with all Fourier coefficients close to zero with probability close to 1/8.

   Analyze the following test: Pick $x, y, z \in \mathbb{Z}_2^n$ and accept if all three of the following conditions hold: (i) $f(x) + f(y) = f(x+y)$, (ii) $f(x) + f(z) = f(x+z)$, and (iii) $f(y) + f(z) = f(y+z)$.

   Prove that the test satisfies our goals, i.e., that it accepts linear functions while accepting functions with largest absolute value of Fourier coefficient being ϵ with probability at most $1/8 + O(\epsilon)$.

   (Hint: As usual let $g(x) = (-1)^{f(x)}$. Fix x, y, z and write the acceptance probability of the test as polynomial in w_1, w_2, and w_3 where $w_1 = g(x)g(y)g(x+y)$, $w_2 = g(x)g(z)g(x+z)$ and $w_3 = g(y)g(z)g(y+z)$.)

4. (Hard, Optional) Generalize the above to a test that makes $k + \binom{k}{2}$ queries and accepts functions that are far from linear with probability roughly $1/2^{\binom{k}{2}}$.