Lecture 2, part 3/3

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We will consider functions $f : \mathbb{F}_2^n \to \{0,1\}$. To simplify notation let us denote $G = \mathbb{F}_2^n$ and $\phi = (1,1,\ldots,1) \in G$.

Definition 1 *We say that a function* $f : G \to \{0,1\}$ has a triangle *if* $\exists_{x,y \in G}\ f(x) = f(y) = f(x+y) = 1$.

Let us denote $\mathcal{F} = \{f : G \to \{0,1\} \mid f \text{ is triangle-free}\}$.

We want to design a probabilistic test that checks if a given function f is triangle-free. The test is as follows: We pick $x, y \in G$ at random and accept if at least one of the values $f(x), f(y), f(x+y)$ is 0.

If f is triangle-free then the test accepts with probability 1. We will show that if f is ϵ-far from being triangle-free then the test rejects with probability at least δ (where δ is some positive function of ϵ).

To prove the above statement we will use Green's regularity lemma.

Definition 2 *We say that a function* $f : G \to \{0,1\} \subset \mathbb{R}$ *is* ϵ-regular *if* $\forall_{\alpha \in G \setminus \{\phi\}}\ |\hat{f}_\alpha| < \epsilon$.

Definition 3 *Let* $f : G \to \{0,1\} \subset \mathbb{R}$. *For a fixed element* $x \in G$ *and a subgroup* H *of* G *we define a function* $f_H^{+x} : H \to \{0,1\}$ *as follows:* $\forall_{y \in H}\ f_H^{+x}(y) = f(x+y)$.

We say that f *is* (H, x, ϵ)-regular *if the function* f_H^{+x} *is* ϵ*-regular.*

Lemma 1 (Green's regularity lemma) *For every* $\epsilon > 0$ *there exists a constant* $c = c(\epsilon)$ *such that the following holds:*

For every function $f : G \to \{0,1\}$ *there exists a subgroup* H *of* G *that satisfies:*
1) $\frac{|G|}{|H|} \leq c$
2) $Pr_x[f \text{ is not } (H,x,\epsilon)\text{-regular}] \leq \epsilon$

Suppose we want to test whether a given function $f : G \to \{0,1\}$ is triangle-free. Let us fix some constant $\epsilon > 0$ and let H be a subgroup as in the lemma.

Group G splits into cosets with respect to H: elements $x, y \in G$ are in the same coset iff $y - x \in H$. If f is (H, x, ϵ)-regular, then it is also (H, x', ϵ)-regular for all elements x' from the same coset as x. From Green's regularity lemma we get that f can be (H, x, ϵ)-not-regular only for elements in ϵ fraction of cosets.

We will create a function $\tilde{f} : G \to \{0,1\}$ which will be close to f and which will have some nice properties. In all the cosets where f is not (H, x, ϵ)-regular the value of $\tilde{f}$ will be 0. In the cosets where 1-s have very small density (in this case $\le 2\epsilon^{1/3}$) $\tilde{f}$ has also values 0 for all elements. In all the other cosets $\tilde{f}$ has the same value as f.

The function $\tilde{f}$ is not very far from f: $\delta(\tilde{f}, f) \le \epsilon + 2\epsilon^{1/3}$.

Let us think what would happen if we run triangle-free test on $\tilde{f}$ instead of f. If $\tilde{f}(x) = 1$ for some $x \in G$ then also $f(x) = 1$. Therefore the rejection probability for f is at least as big as the rejection probability for $\tilde{f}$.

Suppose that $\tilde{f}$ has a triangle (if $dist(f, \mathcal{F})$ is big enough then $\tilde{f}$ must have a triangle). We will show that the test rejects f with some positive probability (which depends on the fixed ϵ). The test rejects if $f(x)f(y)f(x+y) = 1$.

$$Pr[\text{test rejects for } f] = E_{x,y \in G}[f(x)f(y)f(x+y)] = \sum_{\alpha \in G} \hat{f_\alpha}^3$$

If f is ϵ-regular then $Pr[\text{test rejects for } f] \ge \hat{f_\phi}^3 - \epsilon$.

Let us denote by x_0, y_0 the elements for which the triple $(x_0, y_0, x_0 + y_0)$ is a triangle for $\tilde{f}$. We know that in the cosets where x_0, y_0 and $x_0 + y_0$ belong the density of 1-s is not very small (at least $2\epsilon^{1/3}$ fraction of values is 1). We also know that functions $f_H^{+x_0}$, $f_H^{+y_0}$ and $f_H^{+(x_0+y_0)}$ are ϵ-regular. Therefore we get:

$$E_{x,y \in H}[f_H^{+x_0}(x) f_H^{+y_0}(y) f_H^{+(x_0+y_0)}(x+y)] = \sum_{\alpha \in G} \hat{f}_\alpha^{+x_0} \hat{f}_\alpha^{+y_0} \hat{f}_\alpha^{+(x_0+y_0)} \ge$$

$$\ge \hat{f}_\phi^{+x_0} \hat{f}_\phi^{+y_0} \hat{f}_\phi^{+(x_0+y_0)} - \max_{\alpha \neq \phi}\{|\hat{f}_\alpha^{+x_0}|\} \ge 8\epsilon - \epsilon = 7\epsilon$$

There are at most c H-cosets (where $c = c(\epsilon)$ is taken from Green's regularity lemma). For random $x, y \in G$ element x is in the same coset as x_0 and y is in the same coset as y_0 with probability at least $\frac{1}{c^2}$. Therefore we get:

$$Pr[\text{test rejects for } f] = E_{x,y\in G}[f(x)f(y)f(x+y)] \geq$$
$$\geq \frac{1}{c^2} E_{x,y\in H}[f_H^{+x_0}(x) f_H^{+y_0}(y) f_H^{+(x_0+y_0)}(x+y)] \geq \frac{1}{c^2} \cdot 7\epsilon$$

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We have proved that for any $\epsilon > 0$ if $dist(f, \mathcal{F}) > \epsilon + 2\epsilon^{1/3}$ then the test rejects f with probability at least $\frac{1}{c^2} \cdot 7\epsilon$ for some $c = c(\epsilon)$.

At the end we state another regularity lemma:

Lemma 2 (Szemeredi regularity lemma) *For every $\epsilon > 0$ there exists some constant c such that the following holds for every graph G:*

There is a partition $V_1, \ldots, V_c$ of the vertices of G such that all but ϵ-fraction of pairs (V_i, V_j) satisfy the following condition: for all $A \subseteq V_i$ and $B \subseteq V_j$

$$p_{ij}|A||B| - \epsilon|V_i||V_j| \leq E(A \leftrightarrow B) \leq p_{ij}|A||B| + \epsilon|V_i||V_j|$$

where $p_{ij} = \frac{E(V_i \leftrightarrow V_j)}{|V_i||V_j|}$.