Computability Theory, Set Theory and Geometric Measure Theory

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Opening Remarks

The Lebesgue Measure of a set of real numbers A is a numerical indication of the size of A, which can be identified with the integral of the characteristic function of A. The Hausdorff Dimension of a set of real numbers A is a numerical indication of the geometric fullness of A. Sets of positive measure have dimension 1, but there are null sets of every possible dimension between 0 and 1.

Effective Randomness and Effective Hausdorff Dimension are variants which incorporate computability-theoretic considerations. There are direct connections between the effective randomness/dimension of the elements of A and the measure/dimension of A.

The Mathematical Perspective of Computability Theory

Computability theoretic perspective:

- Calibration of the descriptive complexity of mathematical objects and their properties.
- Methods to directly manipulate this complexity, with an emphasis on constructions.
- Minimalism: realize aspects of this complexity generically.

Broadly applicable.

- Calibrate mathematical methods originating elsewhere.
- Include analysis of the representations of mathematical objects when investigating their properties.
- Combine computability theoretic methods with those developed within other mathematical traditions.

Outline

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Lebesgue Measure

For convenience we will work in Cantor space $\mathcal{C}$, wherein the points are infinite binary sequences $x \in 2^\omega$ and a basic open set $B(\sigma)$ consists of all extensions of a particular finite binary sequence $\sigma \in 2^{<\omega}$.

We obtain Lebesgue measure λ on $\mathcal{C}$ by setting $\lambda(B(\sigma)) = 1/2^{|\sigma|}$, where $|\sigma|$ denotes the length of σ, and extend to the σ-algebra of measurable sets. Then, when A is measurable,

$$\lambda(A) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(B(\sigma_k)) : \begin{array}{l} (\sigma_k)_{k\in\mathbb{N}} \text{ is a sequence from } 2^{<\omega} \\ \text{with } A \subseteq \bigcup_{k=1}^{\infty} B(\sigma_k) \end{array} \right\}.$$

Regularity of Lebesgue Measure

Remark

If A is measurable, then

$$\lambda(A) = \inf\{\lambda(O) : O \text{ is open and } A \subseteq O\}$$
$$= \sup\{\lambda(C) : C \text{ is closed and } C \subseteq A\}$$

In other words, the measure of A is carried by the measures of its closed subsets.

The Arithmetic Hierarchy

Definition

In the following A is a subset of ω, for some k.

- A is Σ^0_1 iff there is a computable relation R such that for all x,

$$x \in A \Leftrightarrow (\exists y \in \omega) R(x, y).$$

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- For $n > 1$, A is Σ^0_{n+1} iff there is a Π^0_n relation P such that for all x,

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- Note, $n \in R^X$ can be viewed as defining a $\Sigma^0_1(X)$ subset of ω using R or as defining a Σ^0_1 subset of 2^ω using R and n (a Σ^0_1-class).

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- Note, $n \in R^X$ can be viewed as defining a $\Sigma^0_1(X)$ subset of ω using R or as defining a Σ^0_1 subset of 2^ω using R and n (a Σ^0_1-class).
- Similarly, we can define subsets of 2^ω relative to X, and so forth.

Correspondences

- For $A \subseteq \omega$, A is Σ^0_1 iff A is computably enumerable.

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- For $A \subseteq \omega$, A is Σ^0_1 iff A is computably enumerable.
- For $O \subseteq 2^\omega$, there is an X such that O is Σ^0_1 relative to X iff O is open.
  - When X is omitted, O is an effectively presented open set. Another way to say it is that O is the union of a computably enumerable family of basic open sets.
  - A Π^0_1-class is an effectively presented closed set.

Randomness

formulated by measure

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Definition

A sequence x is *Martin-Löf random* iff it does not belong to any effectively-presented null G_δ set. Precisely, if $(O_k : k \in \mathbb{N})$ is a uniformly computably enumerable sequence of basic open sets such that for all k, O_k has measure less than $1/2^k$, then $x \notin \cap_{k \in \mathbb{N}} O_k$.

Randomness

formulated by compressibility

Definition

- For $\sigma \in 2^{<\omega}$, $K(\sigma)$ is the length of the shortest program which outputs σ and then halts, in a universal prefix-free listing of programs.
- A sequence $x \in 2^{\omega}$ is *algorithmically incompressible* iff there is a C such that for all ℓ, $K(x \upharpoonright \ell) > \ell - C$, where K denotes prefix-free Kolmogorov complexity.

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Theorem (Schnorr 1973)

x is Martin-Löf random iff it is algorithmically incompressible.

Random Sequences and Closed Sets

A closed set C in 2^ω can be represented as the set of infinite paths in a subtree T of $2^{<\omega}$. (The terminal nodes of T index the basic open sets that constitute the complement of C.)

When T is computable, then C is a Π^0_1 class. Otherwise, C is Π^0_1 relative to T.

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When T is computable, then C is a Π^0_1 class. Otherwise, C is Π^0_1 relative to T.

Theorem (Folklore, I first heard it from Yu Liang)

- *If C is Π^0_1 relative to T, then C has positive measure iff C has an element which is Martin-Löf random relative to T.*
- *An arbitrary set A has positive measure iff for all T there is an element of A which is Martin-Löf random relative to T.*

Point-to-Set for Measure

An expository device for what is to follow

Let A denote an arbitrary subset of 2^ω.

For every real, A has an element which is Martin-Löf random relative to that real iff A has positive measure.

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Let A denote an arbitrary subset of 2^ω.

For every real, A has an element which is Martin-Löf random relative to that real iff A has positive measure.

- One proof is to use the regularity of measure and invoke the result on the previous slide.
- Alternatively, one can return to the definition of measure: If A has measure zero, then for every $\epsilon > 0$, there is an open cover of A of measure less than ϵ. Take a real uniformly coding a sequence of covers measures less than $1/2^n$. No element of A is Martin-Löf random relative to that real. The other direction is equally clear.

Outline

Computability Theory and Diophantine Approximation

Diophantine Approximation. Deals with the approximation of real numbers by rational numbers.

Computability Theory. Deals with approximation of real numbers by computation, or more generally by definition.

Both areas study properties of real numbers that are expressed in terms of their representations. There is a natural motivational affinity between them.

Émile Borel (1909): Normal Numbers

Definition

Let ξ be a real number.

- ξ is *simply normal to base b* iff in its base-b expansion, $(\xi)_b$, each digit appears with limiting frequency equal to $1/b$.
- ξ is *normal to base b* iff in $(\xi)_b$ every finite block pattern of digits occurs with limiting frequency equal to the expected value $1/b^\ell$, where ℓ is the block length.
- ξ is *absolutely normal* iff it is normal to every base b.

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Exercise

ξ is normal to base b iff the sequence $(b^n\xi \pmod 1 : n \in \mathbb{N})$ is uniformly distributed in the unit interval.

Normality

If the sequence of digits in $(\xi)_b$ were chosen independently at random, then the simple normality of ξ in base b would be a special case of the Law of Large Numbers.

Theorem (Borel 1909)

Almost all real numbers are absolutely normal.

Problem (Borel)

Give one example of an absolutely normal number.

It is not known whether any (or all) of the familiar irrational numbers are absolutely normal: π, e, $\frac{1+\sqrt{5}}{2}$

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Examples

First constructions of absolutely normal numbers by Lebesgue and Sierpiński, independently, 1917.

Theorem (Champernowne 1933)

$0.123456789101112131415161718192021222324\ldots$ *is normal to base ten.*

An elementary but intricate counting argument shows that Champernowne's number is normal to base 10, but it is not known whether it is absolutely normal.

A computable example

Theorem (Turing ∼1938 (see Becher, Figueira and Picchi 2007))

There is a computable absolutely normal number.

Other computable instances Schmidt 1961/1962, Becher and Figueira 2002.

Absolutely normal numbers in just above quadratic time

Theorem (Becher, Heiber and Slaman 2013)

There is an absolutely normal number ξ such that for any base b, the first n digits in $(\xi)_b$ are computable in essentially n^2-time.

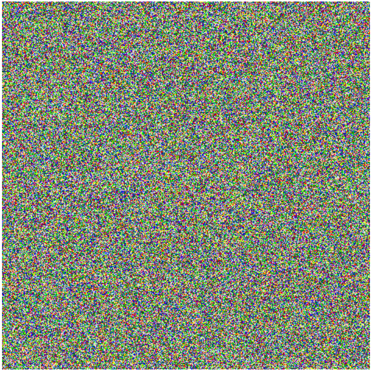
- The proof uses the homogeneity of Lebesgue measure. For a random element of an arbitrary interval, the digits beyond those determined by the interval go to normal at a rate which is independent of the interval.
- Lutz and Mayordomo (2013) and Figueira and Nies (2013) have another argument for an absolutely normal number in polynomial time, based on polynomial-time martingales.

Proof Sketch

The output of the effective algorithm in base 10

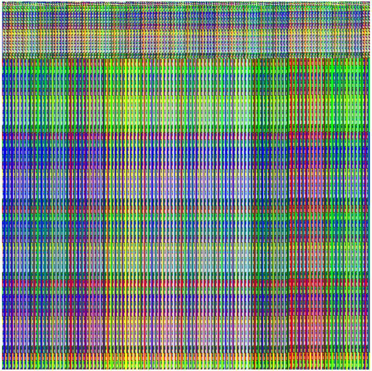
Programmed by Martin Epszteyn

0.40312905420038091323714283808270591027651167776241897751108966366...



First 250000 digits output by the algorithm

Plotted in 500x500 pixels, 10 colors



First 250000 digits of Champernowne

Plotted in 500x500 pixels, 10 colors

Normality to Different Bases

There is one readily-identified connection between normality to one base and normality to another.

Definition

For natural numbers b_1 and b_2 greater than 0, we say that b_1 and b_2 are *multiplicatively dependent* if they have a common power.

Theorem (Maxfield 1953)

If b_1 and b_2 are multiplicatively dependent bases, then, for any real ξ, ξ is normal to base b_1 iff it is normal to base b_2.

Multiplicative independence

Theorem (Schmidt 1961/62)

Let R be a subset of the natural numbers greater than or equal to 2 which is closed under multiplicative dependence. There is a real ξ such that ξ is normal to every base in R and not normal to any base in the complement of R.

Normal numbers and Weyl's criterion

Theorem (Weyl's Criterion)

A sequence $(\xi_n : n \geq 1)$ is uniformly distributed modulo one iff for every complex-valued 1-periodic continuous function f,
$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\xi_n) = \int_0^1 f(x)dx.$
That is, iff for every non-zero integer t, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i t\xi_n} = 0$

Thus, ξ is normal to base b iff for every non-zero t

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i t b^n \xi} = 0.$$

Schmidt's argument rested upon subtle estimates of such harmonic sums.

Arithmetic Definability as a Calibration Tool

Definition

For $A \subseteq \omega$, A is *Σ^0_k-complete* iff the following conditions hold.

- A is Σ^0_k.
- For every Σ^0_k set $B \subseteq \omega$, there is a computable $f : \omega \to \omega$ such that for all m, $m \in B$ iff $f(m) \in A$.
  - We say that A is Σ^0_k-hard

We define Π^0_k-complete similarly.

Arithmetic Definability as a Calibration Tool

Definition

For $A \subseteq \omega$, A is *Σ_k^0-complete* iff the following conditions hold.

- A is Σ_k^0.
- For every Σ_k^0 set $B \subseteq \omega$, there is a computable $f : \omega \rightarrow \omega$ such that for all m, $m \in B$ iff $f(m) \in A$.
    - We say that A is Σ_k^0-hard

We define Π_k^0-complete similarly.

Remark

If A is Π_k^0-complete then A is not Σ_k^0.

Computability Theoretic Independence Between Bases.

Let $\mathcal{S}$ be the set of minimal representatives of the multiplicative dependence classes.

Theorem (Becher and Slaman 2013)

Let R be a Π^0_3 subset of $\mathcal{S}$. There is a real ξ such that ξ is normal to every base in R and not normal to any of the other elements of $\mathcal{S}$. Furthermore, ξ is computable uniformly in the Π^0_3 formula which defines R.

An index set calculation:

Theorem (Becher and Slaman 2013)

The set of real numbers that are normal to at least one base is Σ^0_4-complete.

A fixed point:

Theorem (Becher and Slaman 2013)

For any Π^0_3 formula φ there is a computable real ξ such that for all b in $\mathcal{S}$, ξ is normal to base b iff $\varphi(\xi, b)$ is true.

Constructions

For each of these results, we define a real number ξ by constructing its expansions in different bases.

- Organize by recursion on stages, these are divided into phases according to which base b is receiving attention.
- At each stage, find an appropriate extension of the digits specified from the previous stage by looking at random elements of an appropriate Cantor-like fractal.
  - During a base b phase, show that there is an appropriate extension by using that the properties of normality for random elements of the fractal are invariant under choice of interval, i.e. for invariant under multiplication by b (mod 1).

Proof Sketch

Simple Normality

ξ is *simply normal to base b* iff each digit appears with limiting frequency equal to $1/b$ in the base-b expansion of ξ.

Necessary Conditions:

Theorem

For any base b and real number ξ, the following hold.

- *For any positive integers k and m, if ξ is simply normal to base b^{km} then ξ is simply normal to base b^m.*
- *(Long 1957) If there are infinitely many positive integers m such that ξ is simply normal to base b^m, then ξ is simply normal to all powers of b.*

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Necessary and Sufficient Conditions:

Theorem (Becher, Bugeaud and Slaman 2013)

Let M be a set of natural numbers greater than or equal to 2 such that the following necessary conditions hold.

- *For any b and positive integers k and m, if $b^{km} \in M$ then then $b^m \in M$.*
- *For any b, if there are infinitely many positive integers m such that $b^m \in M$, then all powers of b belong to M.*

There is a real number ξ such that for every base b, ξ is simply normal to base b iff $b \in M$.

Comments on the Proof

We exhibit a Cantor-like construction of a fractal with the following properties.

- Uniform measure concentrates on reals of the desired simple-normality type.
- Hausdorff dimension can be made arbitrarily close to one. We will say more about dimension shortly.
- Self-similarity
  - Not globally self-similar. The splitting levels of the fractal have subintervals with b-adic endpoints and the value of b depends on during which that splitting level was determined.
  - Locally self-similar, as described earlier.

Proof Sketch

Irrationality Exponents

Diophantine version of Kolmogorov complexity

Definition (originating with Liouville 1855)

For a real number ξ, the *irrationality exponent of ξ* is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

- When z is large and $0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$, then p/q is a good approximation to ξ when seen in the scale of $1/q$.
- The irrationality exponent of ξ is a indicator for how well ξ can be approximated by rational numbers (a linear version of Kolmogorov complexity).

Context

- Liouville numbers are those with infinite irrationality exponent.
  - Example: $\sum_{n=1}^{\infty} 1/10^{n!}$
- Almost all real numbers have irrationality exponent equal to 2. In fact, every Martin-Löf random real has irrationality exponent equal to 2.
- (Roth 1955) Irrational algebraic numbers have irrationality exponent equal to 2.

One could cite the second item as natural motivation to refine the notions of size and of randomness. Some subsets of the real numbers are more null than others and some real numbers are more non-random than others.

Outline

Hausdorff Dimension

Define a family of outer measures, parameterized by $d \in [0,1]$. For $A \subseteq 2^\omega$,

$$\mathcal{H}^d(A) = \lim_{r \to 0} \inf \left\{ \sum_i \frac{1}{2^{|\sigma_i|\, d}} : \begin{array}{l} \text{there is a cover of } A \text{ by balls} \\ B(\sigma_i) \text{ with } 1/2^{|\sigma_i|} < r \end{array} \right\}.$$

Definition

The *Hausdorff dimension* of A is as follows.

$$\begin{aligned} \dim_{\mathrm{H}}(A) &= \inf\{d \geq 0 : \mathcal{H}^d(A) = 0\} \\ &= \sup\left(\{d \geq 0 : \mathcal{H}^d(A) = \infty\} \cup \{0\}\right) \end{aligned}$$

Analytic Sets

The condition on the sizes of the basic open sets in the definition of Hausdorff measures makes the analysis of their properties more complicated than those of Lebesgue measure. Similarly, there are restrictions on the families of sets A on which one can prove that these measures are well-behaved.

Analytic Sets

The condition on the sizes of the basic open sets in the definition of Hausdorff measures makes the analysis of their properties more complicated than those of Lebesgue measure. Similarly, there are restrictions on the families of sets A on which one can prove that these measures are well-behaved.

Definition

$A \subset 2^\omega$ is *analytic* iff A is the continuous image of a Borel subset of a Polish space, i.e. a separable completely metrizable topological space.

- Equivalently, $A \subseteq 2^\omega$ is analytic iff it is the projection of a closed set in the cartesian product of 2^ω with the Baire space, ω^ω.

Typically, Hausdorff measures are well-behaved on analytic sets. Proving so can involve an intricate study of the interplay between the definition of measure and the definition of the set being measured.

Frostman's Lemma

Theorem (Frostman 1935, Besicovitch and Davies 1952 (independently))

For A an analytic subset of 2^ω,

$$\dim_H(A) = \sup\left\{ s : \begin{array}{l} \text{there is a Borel measure } \mu \text{ such that } \mu(A) > 0 \\ \text{and for all } \sigma \in 2^{<\omega},\ \mu(B(\sigma)) \leq \left(\frac{1}{2^{|\sigma|}}\right)^s \end{array} \right\}$$

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When the above condition holds for μ, we say that μ is *s-regular* or that μ has the *Mass Distribution Property for s.*

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When the above condition holds for μ, we say that μ is *s-regular* or that μ has the *Mass Distribution Property for s*.

Corollary

If A is an analytic subset of 2^ω and $\dim_H(A) = d$, then for every $s < d$ there is a closed set $C_s \subseteq A$ such that $s \leq \dim_H(C) \leq d$.

In other words, the Hausdorff dimension of analytic A is carried by the dimensions of its closed subsets.

The Jarník-Besicovitch Theorem

Theorem (Jarník 1929 and Besicovitch 1934)

For every real number a greater than or equal to 2, the set of numbers with irrationality exponent equal to a has Hausdorff dimension $2/a$.

By direct application of the definitions, the Hausdorff dimension of the set of numbers with irrationality exponent a is less than or equal to $2/a$. The other inequality comes from an early application of fractal geometry.

Jarník's Fractal

For each real number a greater than 2, Jarník gave a Cantor-like construction of a fractal J contained in $[0, 1]$ of Hausdorff dimension $2/a$ such that the uniform measure ν on J satisfies the following:

- Every element of J has irrationality exponent greater than or equal to a.
- For all b greater than a, the set of numbers with irrationality exponent greater than or equal to b has ν-measure equal to 0.

Effective Hausdorff Dimension

formulated by measure

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Definition

- For $A \subseteq 2^\omega$, define A has *effective s-dimension Hausdorff measure* 0 iff there is a uniformly computably enumerable sequence of open sets $O_i = \bigcup_j B(\sigma_{i,j})$ such that for each i, $A \subseteq O_i$ and $\sum_j (1/2^{|\sigma_{i,j}|})^s < 1/2^i$.
- The *effective Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathit{eff}}(A)$ of A is the infimum of those s such that A has effective s-dimension Hausdorff measure 0.

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- The *effective Hausdorff dimension* $\dim_H^{eff}(A)$ of A is the infimum of those s such that A has effective s-dimension Hausdorff measure 0.

Remark

- *For all A,* $\dim_H(A) \leq \dim_H^{eff}(A)$
- *If x is Martin-Löf random then* $\dim_H^{eff}(\{x\}) = 1$.

Effective Hausdorff Dimension

formulated by compressibility

Definition

A sequence $x \in 2^\omega$ is *algorithmically compressible by a factor of s* iff there is a C such that there are infinitely many ℓ such that $K(x \upharpoonright \ell) \leq s\,\ell - C$, where K denotes prefix-free Kolmogorov complexity.

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Theorem (Mayordomo 2002)

For any $x \in 2^\omega$, $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$ is the infimum of the s such that x is algorithmically compressible by a factor of s.

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Theorem (Mayordomo 2002)

For any $x \in 2^\omega$, $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$ is the infimum of the s such that x is algorithmically compressible by a factor of s.

- We will abbreviate and write $\dim_{\mathrm{H}}^{\mathit{eff}}(x)$ for $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$.
- We can relativize to a real z and write $\dim_{\mathrm{H}}^{\mathit{eff}(z)}(x)$.

Frostman's Lemma Revisited

Theorem (Reimann 2008)

Suppose that $\dim_{\mathrm{H}}^{eff}(x) = d$. *For all* $s < d$, *there is an* s*-regular Borel measure* μ *such that* x *is Martin-Löf random for the measure* μ.

Point-to-Set for Hausdorff Dimension

Theorem (J. Lutz and N. Lutz 2017)

For $A \subseteq 2^\omega$, the Hausdorff dimension of a set A is equal to
the infimum over all $B \subseteq \mathbb{N}$
of the supremum over all $x \in A$
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Notice that there is no restriction on A in the above theorem.

- Unlike in the discussion of measure, since we are not assuming that A is analytic, we cannot immediately move to a closed subset of A.
- We can return to the definition of Hausdorff dimension in terms of open covers. One direction of the above is obtained by considering the reals that can compute appropriate open covers of A. The other direction can be proven by using the relativized version of the fact that the set of reals with effective Hausdorff dimension s has Hausdorff dimension s (as can be derived from the Jarník-Besicovitch Theorem).

Irrationality Exponents

If the irrationality exponent of ξ is equal to a, then ξ has effective Hausdorff dimension less than or equal to $2/a$.

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Theorem (Becher, Reimann and Slaman)

For every $a \geq 2$ and every b in $[0, 2/a]$, there is a real number ξ such that ξ has irrationality exponent a and effective Hausdorff dimension b.

Normality Again

Definition

Suppose μ is a measure on $\mathbb{R}$. The *Fourier Transform* $\hat{\mu}$ of μ is given by

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{2\pi i t\xi} d\mu(\xi)$$

Theorem (Davenport, Erdős and LeVeque 1963)

If μ is a measure on $\mathbb{R}$ such that $\hat{\mu}$ vanishes at ∞ sufficiently quickly, then μ-almost every real number is absolutely normal.

Absolutely Normal Liouville Numbers

- (Kaufman 1981) For any real number $a > 2$, there is a measure ν on the Jarník fractal for a such that the Fourier transform of ν vanishes at infinity. The measure ν is a smooth version of the uniform measure.
- (Bluhm 2000) There is a measure ν supported by the Liouville numbers such that the Fourier transform of ν vanishes at infinity.
- (Bugeaud 2002) There is an absolutely normal Liouville number.

Computing Absolutely Normal Liouville Numbers

Theorem (Becher, Heiber and Slaman 2013)

There is a computable absolutely normal Liouville number.

Again, the proof uses the invariance properties. In this case, it uses invariance in the way that the Fourier transforms of smoothings of Kaufman's measure vanish at infinity.

Irrationality Exponents Relative to Independent Bases

As with normality, the integer bases provide a one-parameter family of compressibility criteria.

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Definition (following Amou and Bugeaud (2010))

For a real number ξ, the *base-b irrationality exponent of ξ* is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{b^k}\right| < \frac{1}{\left(b^k\right)^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

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is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

- If ξ is simply normal in base b, then the base-b irrationality exponent of ξ is equal to 1.

Irrationality Exponent: Depends on Base

Theorem (Amou and Bugeaud (2010))

Suppose that b_1 and b_2 are multiplicative independent bases, and suppose that a_2 and a_3 are greater than $1 + \frac{1+\sqrt{5}}{2}$. There is a real number whose base-b_1 and base-b_2 exponents of irrationality are a_2 and a_3, respectively.

- The proof relies on the theory of continued fractions. A measure theoretic approach would be welcome.

Random or Compressible: Depends on Base

Theorem

There is a real number ξ which is normal to base 2 and whose base 10 exponent of irrationality is equal to ∞.

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The example given is a variation of Stoneham numbers and does not easily generalize. It is open whether it is possible to give an example for base 2 and base 3.

Outline

Return to Hausdorff Measures and Dimension

Consistency results

We have noted that Hausdorff measures behave well on analytic sets.

Whether this good behavior extends past analytic turns out to be a subtle question, whose answer depends on meta-mathematical considerations.

The Projective Hierarchy

Definition

In the following A is a subset of ω, 2^ω, ω^ω or some finite product of these.

- A is Σ^1_1 iff there is a Π^0_1 predicate R such that for all x,

$$x \in A \Leftrightarrow (\exists y \in \omega^\omega) R(x, y).$$

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- A is Σ^1_1 iff there is a Π^0_1 predicate R such that for all x,
$$x \in A \Leftrightarrow (\exists y \in \omega^\omega) R(x, y).$$
- For $n \geq 1$, A is Π^1_n iff the complement of A is Σ^1_n.
- For $n > 1$, A is Σ^0_{n+1} iff there is a Π^0_n relation B such that for all x,
$$x \in A \Leftrightarrow (\exists y) B(x, y).$$

Correspondences

- For A a subset of 2^ω or ω^ω, there is an $X \in 2^\omega$ such that A is Σ^1_1 relative to X iff A is analytic.
  - Similarly for Π^1_1 and co-analytic.
- Sets that are Σ^1_1/Π^1_1 (without any X) are effectively presented analytic/co-analytic sets.

Gödel's Universe of Constructible Sets

In his proof of the consistency of the Axiom of Choice and the Continuum Hypothesis, Gödel defined L, the class of *constructible sets*.

$$
\begin{aligned}
L_0 &= \emptyset \\
L_{\alpha+1} &= \{x : x \text{ is first-order definable in } (L_\alpha, \in) \text{ using parameters}\} \\
L_\lambda &= \bigcup_{\beta<\lambda} L_\beta
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$$

- A set x is in L iff there is an ordinal β such that $x \in L_\beta$.
  - We say that *x is constructed at β* when β is the least ordinal such that $x \in L_\beta$.
- Gödel's Theorems
  - $V = L$, the assertion that every set is constructible, is consistent with ZFC.
  - $V = L$ implies that there is a Σ^1_2 wellordering of 2^ω, based on the order of constructibility.

Co-analytic Sets

Definition

Define P by

$$P = \left\{ X : \begin{array}{l} X \text{ can compute a representation of the ordinal at} \\ \text{which } X \text{ is constructed} \end{array} \right\}$$

Theorem (Gaspari, Kechris, Sacks)

P is the maximal thin Π^1_1 set.

- *X belongs to P iff X can compute a structure which satisfies the inductive condition of being an initial segment of L and which is well-founded. This is a Π^1_1 property.*
- *P has no perfect subset.*
- *If $V = L$ then P is not countable. In fact, if $V = L$ then every $Y \in 2^\omega$ is computable relative to some $X \in P$.*

Co-analytic Sets

Working in $V = L$

Theorem

If $V = L$ *then* $\dim_H(P) = 1$.

Co-analytic Sets

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If $V = L$ *then* $\dim_H(P) = 1$.

Consequently, it is consistent with *ZFC* that the Hausdorff dimensions of co-analytic sets are not carried by their closed subsets.

Applying Point-to-Set Reasoning

Working in $V = L$

We work in L and give a sketch of the proof.

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Step 1. There is an infinite computable set $S \subseteq \mathbb{N}$ such that for all B and for all x, if x is Martin-Löf random relative to B and y is equal to x at all places not in S then $\dim_{\mathrm{H}}^{\mathit{eff}(B)}(y) = 1$.

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In fact, S could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of y relative to B.

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In fact, S could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of y relative to B.

Step 2. By the Lutz and Lutz theorem, it is sufficient to show that for every z there is a y in P such that $\dim_{\mathrm{H}}^{eff(z)}(y) = 1$.

Applying Point-to-Set Reasoning

Working in $V = L$

Step 3. Suppose that $B \in 2^\omega$ is given.

- Let x be Martin-Löf random relative to B.
- Let $m \in P$ be such that m can compute x and B.
- Let y be the result of replacing the bit values of x on the elements of S by the bit values of m.

Then, m can compute the ordinal at which y is constructed and y can compute m. Thus, $y \in P$.

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Then, m can compute the ordinal at which y is constructed and y can compute m. Thus, $y \in P$.

Step 4. Conclude, $\dim_H(P) = 1$, as required.

Capacitability in Models of AD

At the opposite end of the set theoretic spectrum, AD/Large Cardinals show that capacitability extends far beyond that class of analytic sets.

Theorem (Crone, Fishman and Jackson 2020)

Assume AD. Let A be a subset of $\mathbb{R}^d$ and $0 \leq \delta \leq d$. Either A has a compact subset C such that $\dim_{\mathsf{H}}(C) \geq \delta$ or $\dim_{\mathsf{H}}(A) \leq \delta$.

The proof artfully combines ingredients from Descriptive Set Theory and Geometric Measure Theory, in particular a geometric formulation of $\dim_{\mathsf{H}}(A)$ in terms of the measure of A within neighborhoods of elements of A.

A Second Example under $V = L$

projections of sets of positive dimension

Theorem (Marstrand 1952)

Let $E \subseteq \mathbb{R}^2$ be analytic. Then, for almost every angle $\theta \in [0, 2\pi]$,

$$\dim_H(p_\theta E) = \min\{\dim_H(E), 1\},$$

where $p_\theta(x, y) = x\cos\theta + y\sin\theta$. Moreover, if $\dim_H(E) > 1$, then $\mu(p_\theta E) > 0$, for almost every angle θ.

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where $p_\theta(x, y) = x \cos\theta + y \sin\theta$. Moreover, if $\dim_H(E) > 1$, then $\mu(p_\theta\, E) > 0$, for almost every angle θ.

Theorem (Slaman and Stull (in progress))

If $V = L$ then for every $s \in (0, 1)$ there is a Π^1_1 set $E \subseteq \mathbb{R}^2$ such that $\dim_H(E) = 1 + s$ but

$$\dim_H(p_\theta E) = s$$

for every $\theta \in (0, 2\pi)$.

A Third Example: Gauge Functions and General Hausdorff Dimension (joint with Jan Reimann)

Definition

A *gauge function* is a function $h : (0, \infty) \to (0, \infty)$ which has the following properties:

- continuous
- increasing
- $\lim_{t \to 0^+} h(t) = 0$

Example

For $s > 0$, $t \mapsto t^s$ is a gauge function.

Gauge Functions and General Hausdorff Dimension

Definition

Let h be a gauge function. For a set $A \subseteq 2^\omega$ (or ω^ω, $\mathbb{R}^n$ etc.), define

$$H^h(A) = \lim_{\delta \to 0} \inf_{\substack{A \subseteq \cup F_i \\ \max d(F_i) < \delta}} \sum_{i=1}^{\infty} h(d(F_i))$$

where $\{F_i\}$ is a sequence of closed (open) sets covering A and $d(F_i)$ is the diameter of F_i.

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where $\{F_i\}$ is a sequence of closed (open) sets covering A and $d(F_i)$ is the diameter of F_i.

- When $h(t)$ is t^s, $H^h = H^s$ is the s-dimensional Hausdorff outer measure that we have been discussing.
- Gauge functions provide a more finely graded calibration of measure and thereby of dimension than is given by the usual family $\{t \mapsto t^s : s \in (0, 1]\}$.

Gauge Functions and General Hausdorff Dimension

Definition

Write $h \prec g$ to indicate that $\lim_{t \to 0^+} \frac{g(t)}{h(t)} = 0$.

- If $h \prec g$ then it is easier for a set to be H^g-null than it is to be H^h-null.
- Possible source of confusion: If $h \prec g$ then $h(t)$ is larger than $g(t)$ when t is close to 0.

Example

$t^{\log(2)/\log(3)} \prec t^1$, since $\lim_{t \to 0^+} \frac{t}{t^{\log(2)/\log(3)}} = 0$.

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Similarities (motivates the notation $\prec$):

- If $H^h(A)$ is finite and $h \prec g$ then $H^g(A) = 0$.

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- If $H^h(A)$ is not zero and $j \prec h$ then $H^j(A)$ is infinite.

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Difference:

- (Besicovitch 1956) If $H^h(A) = 0$ then there is a j with $j \prec h$ such that $H^j(A) = 0$.

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Definition

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Improved observation from previous slide:

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Example

- For d in $(0, 1)$, the set $D_d^{eff} = \{x : \dim_{\mathrm{H}}^{eff}(x) = d\}$ has Hausdorff dimension d and is non-σ-finite for H^d.
- The set of Liouville numbers is non-σ-finite for every h such that for all $d \in (0, 1]$, $\lim_{t \to 0^+} \frac{t^d}{h(t)} = 0$.

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- The set of Liouville numbers is non-σ-finite for every h such that for all $d \in (0,1]$, $\lim_{t\to 0^+} \frac{t^d}{h(t)} = 0$.

Question

Is there a useful point-to-set formulation of a set's being non-σ-finite for H^h?

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f[f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g[h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

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As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Example

A line segment within the plane has strong dimension 1.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

Thus, if E is compact then E cannot have strong dimension h and be non-σ-finite for H^h.

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Theorem (Davies 1956 for x^s, Sion and Sjerve 1962)

If E is analytic and is non-σ-finite for H^h, then there is a compact subset of E that is non-σ-finite for H^h.

Hence, we can make the above observation for analytic sets.

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Hence, we can make the above observation for analytic sets.

It would be interesting to find proofs of these theorems using effective methods.

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If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

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If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

Theorem (Combining Besicovitch 1963 with Erdős, Kunen and Mauldin 1981)

If $V = L$ there there is a Π^1_1 set $E \subseteq \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

Proof Sketch

$E = A \times [0, 1]$, where A is a small uncountable set.

- (Besicovitch) CH implies that there is an uncountable set A such that any open cover of $\mathbb{Q}$ is an open cover of A.
- (Erdős, Kunen and Mauldin) $V = L$ implies that there is a co-analytic example of such an A.

Borel Conjecture

a connection with Lebesgue measure

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

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Borel (1919) conjectured that strong measure 0 implies countable (BC).

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Theorem

- *(Sierpiński 1928) CH implies that there is an uncountable set of strong measure 0.*
- *(Laver 1976) $Con(ZFC)$ implies $Con(ZFC + BC)$.*

Borel Conjecture

Theorem (Besicovitch 1955)

A set E has strong dimension 0 iff it has strong measure 0.

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Theorem (Besicovitch 1955)

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Theorem (Another variation on Besicovitch 1963)

$\neg BC$ implies that there is a subset of $\mathbb{R}^2$ which has strong linear dimension and which is non-σ-finite for linear measure.

A Challenge

Question

Does the Borel Conjecture imply that there do not exist h and A such that A has strong dimension h and A is not σ-finite for H^h?

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Does the Borel Conjecture imply that there do not exist h and A such that A has strong dimension h and A is not σ-finite for H^h?

The conceptual challenge is to overcome the intractability of the property that A is non-σ-finite for H^h.

Understanding Non-σ-finiteness

A case study

Consider Π^0_1 subsets of $2^\omega \times 2^\omega$ and linear measure H^1.

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Exercise

The set of indices for Π^0_1 subsets C of $2^\omega \times 2^\omega$ such that $H^1(C) \neq 0$ is arithmetic.

By the compactness of $2^\omega \times 2^\omega$, we can assume that all the open covers in the definition of $H^1(C)$ are finite, which means that the prima facie definition of "$H^1(C) \neq 0$" can be expressed arithmetically.

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Let $N\sigma Finite$ be the of indices for Π^0_1 subsets C of $2^\omega \times 2^\omega$ such that C is non-σ-finite for H^1

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$N\sigma Finite$ is Σ^1_1-complete.

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Theorem

$N\sigma Finite$ is Σ^1_1-complete.

Here is a more familiar situation which is analogous.

Exercise

The set of indices for Π^0_1 subsets C of 2^ω such that C is uncountable Σ^1_1-complete.

Use Cantor's theorem: C is uncountable iff C has a perfect subset.

Understanding Non-σ-finiteness

NσFinite is Σ^1_1

The ingredients in the proof of Davies's (1956) theorem about capacitability of non-σ-finiteness entail the following:

> C is non-σ-finite for H^1 iff there is perfect tree of closed sets such that each path corresponds to a closed set of H^1-positive measure.

- It follows that *NσFinite* is a Σ^1_1 set.
- Show that *NσFinite* is Σ^1_1-hard by direct construction, analogous to the analysis of Cantor's theorem.

Understanding Non-σ-finiteness

NσFinite is Σ^1_1

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Remark

We also exhibit a perfect tree of pairwise-disjoint closed subsets of positive measure in the examples that D_d^{eff} is non-σ-finite for H^d and the set of Liouville numbers is non-σ-finite for H^h whenever h is $\prec$-below all the power functions $t \mapsto t^s$.

Thank you for your attention.

The End