Computability Theory, Set Theory and Geometric Measure Theory

Exercises

1. Suppose that A is a Π^0_1 subset of 2^ω.
   (a) Show that there is a computable tree $T \subseteq 2^{<\omega}$ such that A is the set of infinite paths in T.
   (b) Show that if A is finite then all the elements of A are computable.
2. Show that there is a Π^0_1 set A such that A is uncountable and has no computable element.
3. Show that if R is ML-random then R is algorithmically incompressible.
4. Show that $\{X \in 2^\omega : X \text{ is normal }\}$ is a Π^0_3-complete set.
5. Suppose that S is a Σ^0_3 subset of 2^ω such that every normal $X \in 2^\omega$ is an element of S. Show that P has a computable element which is not normal. (Hint: Use the fixed point theorem from the lecture.)
6. Show that if a real number $\xi \in [0, 1]$ has irrationality exponent greater than 2 then the sequence of digits in the binary expansion of ξ is not ML-random.
7. Suppose that $A(x, y) \subseteq \omega^2$ is Σ^1_1 and define

   $$B(x) \text{ iff } (\forall y \in \omega)A(x, y).$$

   Show that $B(x)$ is Σ^1_1.
8. Suppose that $f \prec g$ are gauge functions. Show that there are uncountably many h such that $f \prec h \prec g$.
9. Show that if $H^h(A)$ is finite and $h \prec g$ then $H^g(A) = 0$.
10. Show that the set of indices for Π^0_1 subsets C of 2^ω such that C is uncountable is Σ^1_1-complete.