Structurally tractable graph classes – assignment

Solutions due: 15 September 2021, send to `szymtor@mimuw.edu.pl`

Below are 13 problems, of varying difficulty. Each correct solution to a problem is worth 10 points, so in total 130 points can be attained. The grading scale is as follows:

70 points: pass (3),

90 points: good (4),

110 points: very good (5).

Let $\mathcal{C}$ be a class of logical structures. Say that the model-checking problem is *fixed-parameter tractable* on $\mathcal{C}$ if there is an algorithm that given a first-order sentence φ and an n-element structure from $\mathcal{C}$ decides whether it satisfies φ in time $f(\varphi) \cdot n^c$, for some fixed constant c and computable function f.

Colored total orders

Problem 1.

Fix a number $l \in \mathbb{N}$. A *colored total order* is a set X equipped with a total order $<$ and unary predicates $U_1, \ldots, U_l \subseteq X$. Using the (de)composition method, prove that the model-checking problem on the class of colored total orders is fixed-parameter tractable. ■

Equivalence relations

For a number $k \geqslant 0$, let $\mathcal{C}_k$ be the class of all finite sets equipped with k equivalence relations $\sim_1, \ldots, \sim_k$, and let $\mathcal{C}'_k \subseteq \mathcal{C}_k$ be its subclass consisting of those structures in which the equivalence relations are nested, that is

$$\sim_1 \supseteq \ldots \supseteq \sim_k.$$

Problem 2.

Show that if the model-checking problem is fixed-parameter tractable on $\mathcal{C}_2$, then it is fixed-parameter tractable on the class of all graphs (which is conjectured not to be the case). ■

Problem 3.

Show that the model-checking problem is fixed-parameter tractable on $\mathcal{C}'_k$, for every fixed $k \geqslant 0$, by reducing the problem to the model-checking problem on words, considered in Problem 1. More precisely, show that there is a polynomial-time computable function $f\colon \mathcal{C}'_k \to A^*$, where A is some finite alphabet, and a computable function $\varphi \mapsto \varphi'$ mapping each sentence φ in the signature $\{\sim_1, \ldots, \sim_k\}$ to a sentence φ' in the signature $\{<, (L_a)_{a \in A}\}$ such that

φ holds in S if and only if φ' holds in $f(S)$

for every $S \in \mathcal{C}'_k$ and sentence φ in the signature $\{\sim_1, \ldots, \sim_k\}$. ■

Locality

Let G be a graph and let $E(x, y)$ denote that x and y are adjacent vertices of G. Define the relation $\bar{a} \simeq_k^{\text{loc}} \bar{b}$, where $\bar{a}$ and $\bar{b}$ are tuples of vertices of G of the same length, inductively as follows:

- For $k = 0$, write $\bar{a} \simeq_k^{\text{loc}} \bar{b}$ if $\bar{a}$ and $\bar{b}$ have the same atomic type, that is, satisfy exactly the same atomic formulas $E(x, y)$ and $x = y$.
- For $k \geqslant 1$, write $\bar{a} \simeq_k^{\text{loc}} \bar{b}$ if for every vertex v within distance at most 2^{k-1} from either of the vertices in $\bar{a}$ there is a vertex w such that $\bar{a}v \simeq_{k-1}^{\text{loc}} \bar{b}w$ and conversely, for every vertex w within distance at most 2^k from either of the vertices in $\bar{b}$ there is a vertex v such that $\bar{a}v \simeq_{k-1}^{\text{loc}} \bar{b}w$.

The relation $\simeq_k^{\text{loc}}$ can be equivalently defined in terms of a variant of the usual Ehrenfeucht-Fraïssé game, where in the ith round of the game, Spoiler is required to place his pebble within distance at most 2^{k-i} from one of the pebbles played previously. Finally, $\bar{a} \simeq_k^{\text{loc}} \bar{b}$ if and only if $\bar{a}$ and $\bar{b}$ have the same *local type* of quantifier-rank k, as defined in the lecture.

Problem 4.

Prove that, for all $k \geqslant 0$, if $a_1a_2 \simeq_k^{\text{loc}} b_1b_2$ then either $\text{dist}(a_1, a_2) = \text{dist}(b_1, b_2)$ or $\text{dist}(a_1, a_2)$ and $\text{dist}(b_1, b_2)$ are both larger than 2^k. ■

Problem 5.

Prove that if a, b are two vertices of a finite graph G then $a \simeq_k^{\text{loc}} b$ for all $k \geqslant 0$ if and only if there is an automorphism of G (that is, a permutation of the vertices that preserves adjacency and non-adjacency) that maps a to b and b to a. ■

Problem 6.

Prove that if a, b are two vertices of a graph G such that:

- the distance between a and b is at least 2^k in G, and
- $a \simeq_k^{\text{loc}} b$,

then a and b have equal types of quantifier rank k (that is, satisfy the same formulas $\varphi(x)$ of quantifier rank k, or equivalently, duplicator wins the usual k-round Ehrenfeucht-Fraïssé game, starting from the position with vertices a and b marked). ■

All the above can be lifted to colored graphs, that is, graphs equipped with a fixed number of unary predicates $U_1, \ldots, U_l$. To this end, redefine the base case $a \simeq_0^{\text{loc}} b$ by requiring that $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas $E(x, y)$ and $x = y$ and $U_i(x)$, for $i = 1, \ldots, l$. The remainder of the inductive definition of $\simeq_k^{\text{loc}}$ remains as above, and the statement of the previous problem now holds for colored graphs. Colors can be used in particular to mark specific elements, e.g. a graph G together with two distinguished nodes c, d can be viewed as a colored graph where $U_1 = \{c\}$ and $U_2 = \{d\}$.

Problem 7.

Prove that the model checking problem is fixed-parameter tractable on any class of graphs with bounded maximum degree, using the statement of the previous problem lifted to colored graphs.

Recall that in the lecture, this fact was proved using a variant of the statement in Problem 6, that for every k there is a l such that $a \simeq_l^{\mathrm{loc}} b$ implies that a and b have equal types of quantifier rank k. ■

Twin-width

A matrix M over an alphabet A is a function $M\colon R \times C \to A$, where R and C are two totally ordered sets of *rows* and *columns*, respectively. We say that M is an $m \times n$ matrix if $m = |R|$ and $n = |C|$.

A *submatrix* of M is obtained by picking a subset $R' \subseteq R$ and a subset $C' \subseteq C$ and restricting M to $R' \times C'$. The submatrix is *contiguous* if R' and C' are intervals in R and C, respectively. A matrix M is *mixed* if the function $M\colon R \times C \to A$ depends on both coordinates, that is, there are $r \in R$ and $c, c' \in C$ with $M(r, c) \neq M(r, c')$ and there are $r, r' \in R$ and $c \in C$ with $M(r', c) \neq M(r, c)$. A 2×2 mixed matrix is called a *corner*. Say that M *has a corner* if there is a 2×2 contiguous submatrix of M which is a corner.

Problem 8.

Prove that a matrix M over the alphabet $\{0, 1\}$ is mixed if and only if it M has a corner. ■

Problem 9.

Show that every matrix M over the alphabet $\{0, 1\}$ satisfies at least one of the following conditions:

1. M has a corner, or
2. M is constantly equal to 0, or
3. M has a row full of 1's, or
4. M has a 1 in the last row.

Say that a matrix M over the alphabet $\{0, 1\}$ has a *t-grid minor* if M can be partitioned into t^2 contiguous submatrices by using $t-1$ horizontal and $t-1$ vertical lines, so that each submatrix contains a 1. Similarly, M has a *t-mixed minor* if M can be partitioned into t^2 contiguous submatrices, by using $t-1$ horizontal and $t-1$ vertical lines, so that each submatrix is mixed.

In the following, a simple variant of Ramsey's theorem may be helpful:

Theorem 1. *For every n and C there is some N such that every $N \times N$ matrix over a C-element alphabet contains an $n \times n$ submatrix which is constant.*

Problem 10.

Prove that for every $t, k \in \mathbb{N}$ there is some $c \in \mathbb{N}$ such that the following implication holds. If a matrix M has a c-grid minor then either M has a t-mixed minor or M has some $k \times k$ (not necessarily contiguous) submatrix consisting of 1's. ■

Recall the following characterisation of bounded twin-width.

Theorem 2. *A class $\mathcal{C}$ of graphs has bounded twin-width if and only if there is some constant t such that for every $G \in \mathcal{C}$, there is a total order of the vertices of G such that the adjacency matrix of G along that order has no t-mixed minor.*

Also recall the result of Marcus and Tardos:

Theorem 3. *For every t there is a constant c such that every $n \times n$-matrix M over the alphabet $\{0, 1\}$ with at least $c \cdot n$ many 1's contains a t-grid minor.*

Problem 11.

Using the statement of the previous problem, show that if $\mathcal{C}$ is a class of graphs of bounded twin-width such that there is some biclique $K_{t,t}$ which is not a subgraph of any graph in $\mathcal{C}$, then the graphs in $\mathcal{C}$ have bounded degeneracy (a graph has degeneracy at most d if every its subgraph contains a vertex of degree at most d). ■

Problem 12.

Say that a graph G has *book number* k if there is a total order $<$ of the vertices of G and a partition of the edges of G into k sets such that no two edges vw and $v'w'$ in the same part of the partition cross, that is, satisfy $v < v' < w < w'$.

Using the characterisation of twin-width via mixed minors (Thm. 2), show that if a class of graphs $\mathcal{C}$ has bounded book number then $\mathcal{C}$ has bounded twin-width. ■

Bounded expansion

Let $f \colon \mathbb{N} \to \mathbb{N}$ be such that $f(n) = k$ if 2^k is the highest power of 2 which divides n. The class $\mathcal{C}$ consists of finite subgraphs of the infinite graph G with vertices $\{1, 2, \ldots\} \cup \{c_0, c_1, c_2, \ldots\}$ and edges

$$\{i\text{-}(i+1) \mid i \geqslant 1\} \cup \{i\text{-}c_{f(i)} \mid i \geqslant 1\}.$$

Recall that given a graph G and an order $<$ on its vertices, a vertex v is *weakly r-reachable* from w if there is a path from w to v of length at most r such that v is the $<$-smallest vertex on this path. By $\mathrm{wcol}_r(G, <)$ we denote the maximum, over all vertices w of G, of the number of vertices v that are weakly r-reachable from w.

Also recall that a class $\mathcal{C}$ of graphs has bounded expansion if and only if for every r there is a constant c such that for every $G \in \mathcal{C}$ there is a total order $<$ on $V(G)$ such that $\mathrm{wcol}_r(G, <)$ is at most c.

Problem 13.

Find an ordering $<$ of the vertices of G such that $\mathrm{wcol}_r(G,<)$ is finite, for every $r \in \mathbb{N}$. Conclude that the class $\mathcal{C}$ has bounded expansion. ■