Combinatorial limits

Part 3

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Overview of the course

- Limits of dense graphs

  Survey of main concepts in the area

- The flag algebra method

  Applications in extremal combinatorics

- Limits of sparse graphs

  Various concepts, less understood

Flag algebras

- algebra $\mathcal{A}$ of formal linear combinations of graphs
- homomorphism $f_W : \mathcal{A} \to \mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
  multiplication, other relations between elements
- algebra $\mathcal{A}^R$ of R-rooted graphs
  random homomorphism $f_W^R : \mathcal{A}^R \to \mathbb{R}$
  multiplication, average operator $[\![\cdot]\!]_R : \mathcal{A}^R \to \mathcal{A}$
  $\mathbb{E}_R f_W^R(x) = f_W([\![x]\!]_R)$ for every $x \in \mathcal{A}^R$
- $f_W([\![x^2]\!]_R) \geq 0$ - how to find suitable x?

SDP FORMULATION

- find maximum α_0 such that $f_W(G_0) \geq \alpha_0$ assming $f_W(G_i) \geq \alpha_i$ where $G_0, \ldots, G_k \in \mathcal{A}$
- What inequalities can we use?
  $f_W(G') \geq 0$ for any graph G'
  $f_W(K_1) = 1$ where K_1 expressed in n-vertex graphs
  $f_W(\llbracket x^2 \rrbracket_R) \geq 0$ for $x \in \mathcal{A}^R$
- let $H_1, \ldots, H_m$ be elements of $\mathcal{A}^R$, $h = (H_1, \ldots, H_m)$
  if $M \succeq 0$, then $f_W(\llbracket h^T M h \rrbracket_R) \geq 0$

SDP formulation

- prove $f_W(G_0) \geq \alpha_0$ assuming $f_W(G_i) \geq \alpha_i$
- find $\gamma_i \geq 0$, $\delta_0 \in \mathbb{R}$, $\delta_i \geq 0$, $M \succeq 0$
  $G_0 = \sum_{i=1}^{k} \gamma_i G_i + \sum_{i=1}^{\ell} (\delta_0 + \delta_i) G'_i + \llbracket h^T M h \rrbracket_R$
  $\alpha_0 = \delta_0 + \sum_{i=1}^{k} \gamma_i \alpha_i$
  where $G'_1, \ldots, G'_\ell$ are all n-vert. graphs and $h \in (\mathcal{A}^R)^m$
- $\gamma_i \times f_W(G_i) \geq \gamma_i \times \alpha_i$
  $\delta_0 \times f_W(G'_1 + \cdots + G'_\ell) = \delta_0 \times 1$
  $\delta_i \times f_W(G'_i) \geq 0$
  $f_W(\llbracket h^T M h \rrbracket_R) \geq 0$

SDP example

- prove $f_W(\overline{K_3} + K_3) \geq \alpha_0$ for maximum α_0
- $(G'_1, \ldots, G'_4) = (\overline{K_3}, \overline{K_{1,2}}, K_{1,2}, K_3)$, $h = (\overline{K_2}^{\bullet}, K_2^{\bullet})$
- SDP: $\max\langle C, X\rangle$ s.t. $\langle A_i, X\rangle = b_i$, $X \succeq 0$, $X \in \mathbb{R}^{8\times 8}$

$$C = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad X = \begin{matrix} 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & -3/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3/4 & 3/4 \end{matrix}$$

$$A_1 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad b_1 = 1$$

$$A_2 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{matrix} \qquad b_2 = 0$$

$$A_3 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{matrix} \qquad b_3 = 0$$

$$A_4 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \qquad b_4 = 1$$

SDP FORMULATION

- prove $f_W(G_0) \geq \alpha_0$ if $f_W(G_i) \geq \alpha_i$
- find $\gamma_i \geq 0$, $\delta_0 \in \mathbb{R}$, $\delta_i \geq 0$, $M \succeq 0$
  $G_0 = \sum_{i=1}^{k} \gamma_i G_i + \sum_{i=1}^{\ell} (\delta_0 + \delta_i) G'_i + [\![h^T M h]\!]_R$
  $\alpha_0 = \delta_0 + \sum_{i=1}^{k} \gamma_i \alpha_i$
  where $G'_1, \ldots, G'_\ell$ are all n-vert. graphs and $h \in (\mathcal{A}^R)^m$
- SDP: $\max \langle C, X \rangle$ s.t. $\langle A_i, X \rangle = b_i$ and $X \succeq 0$
  X of size $k + 2 + \ell + m$, diagonal γ_i, $\pm\delta_0$, δ_i, M
  ℓ constraints, b_i is the coefficient of G'_i in G_0

Questions?

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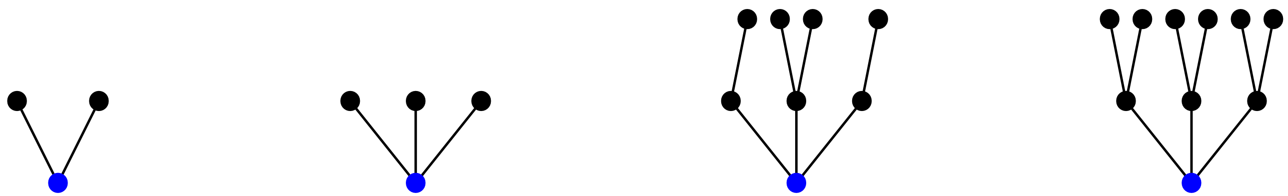
Sparse graph convergence

- convergence of graphs with bounded degree
  trivially converging to the zero graphon
- need of a different notion of convergence
  several notions, each having some cons
- absence of understood analytic representation
  characterization of realizable neighborhood statistics
  Aldous and Lyons Conjecture, relation to group theory

Benjamini-Schramm convergence

- introduced by Benjamini and Schramm in 2001
  also referred to as left convergence
- bounded number of types of d-neighborhoods
  convergence of statistic of d-neighborhoods
- cons: connected vs. disconnected (G vs. $G \cup G$)
  bipartite vs. non-bipartite graphs (random graphs)



Left convergence

- graph homomorphism $\varphi : G \to H$
  for every $uv \in E(G)$, $\varphi(u)\varphi(v) \in E(H)$
- $\hom(G, H)$ = number of homomorphisms from G to H
- Dense graph convergence
  $(G_n)_{n\in\mathbb{N}}$ converges $\Leftrightarrow$ $\frac{\hom(H,G_n)}{|V(G_n)|^{|V(H)|}}$ converges for all H
  equivalent to subgraph densities by PIE
- Benjamini-Schramm convergence
  $(G_n)_{n\in\mathbb{N}}$ converges $\Leftrightarrow$ $\frac{\hom(H,G_n)}{|V(G_n)|}$ converges for conn. H

Local-global convergence

- introduced by Hatami, Lovász and Szegedy in 2012
- types of d-neighborhoods k-vertex-colored graphs
  convergence of d-neighborhood statistics
  attainable by a k-vertex-coloring of graphs
- K = number of k-vertex-colored d-neighborhood types
  $\forall k, d : (G_i)_{i\in\mathbb{N}}$ yields $(A_i)_{i\in\mathbb{N}}$ where $A_i \subseteq \mathbb{R}^K$
  $\forall\varepsilon > 0 \; \exists\, n \; \forall\, i, j > n, x \in A_i \; \exists y \in A_j \; ||x - y|| \leq \varepsilon$
- local-global convergence $\Rightarrow$ left convergence

Graphings

- graphing G is a graph with $V(G) = [0,1]$
  bounded maximum degree, Borel edge-set
  mass preservation: $\int_A \deg_B(x)\,\mathrm{d}x = \int_B \deg_A(y)\,\mathrm{d}y$
  where $\deg_Y(x) = |\{y \text{ s.t. } (x,y) \in G\}|$
- Theorem (Elek, 2007)
  Every BS-convergent sequence has a graphing.
  Theorem (Hatami, Lovász, Szegedy, 2012)
  Every LG-convergent sequence has a graphing.

Questions?

Thank you for your attention!