Combinatorial limits

Part 2

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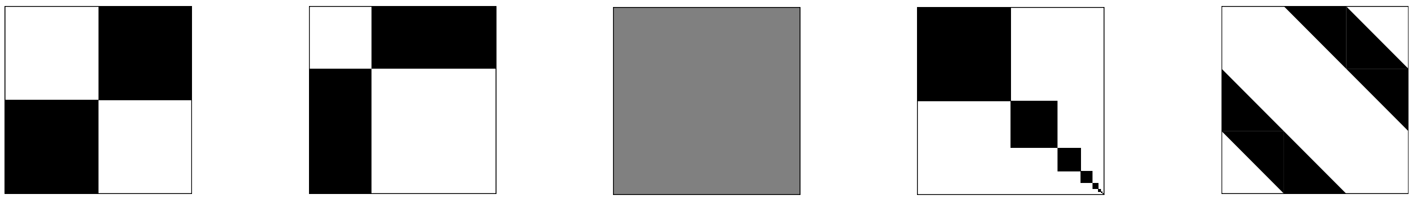
Overview of the course

- Limits of dense graphs
  Survey of main concepts in the area
- The flag algebra method
  Applications in extremal combinatorics
- Limits of sparse graphs
  Various concepts, less understood

Limit object: graphon

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$



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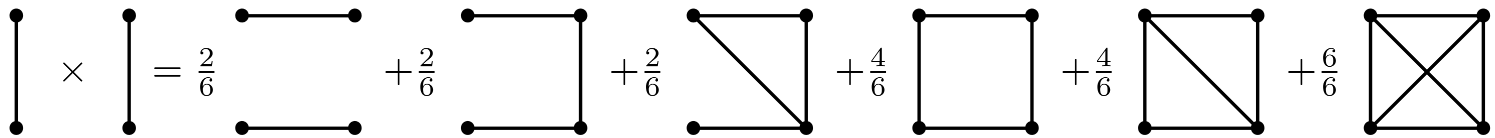
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$
- W-random graphs converge to W with probability one
- every convergent sequence of graphs has a limit

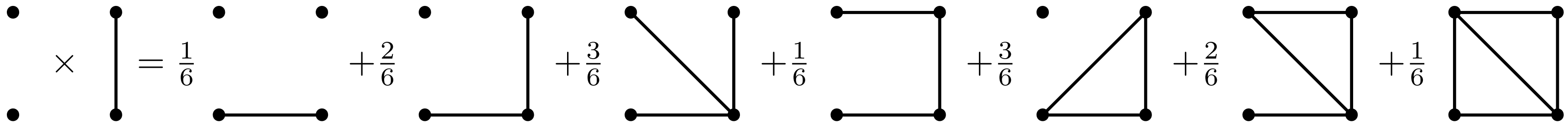
Flag algebras

- the flag algebra method independent of graph limits
  we introduce the method using graphons for simplicity
- algebra $\mathcal{A}$ of formal linear combinations of graphs
  addition and multiplication by a scalar
- homomorphism f_W from $\mathcal{A}$ to $\mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
- examples: $f_W(K_2) = d(K_2, W)$
  $f_W(K_2 - K_3) = d(K_2, W) - d(K_3, W)$

Multiplication

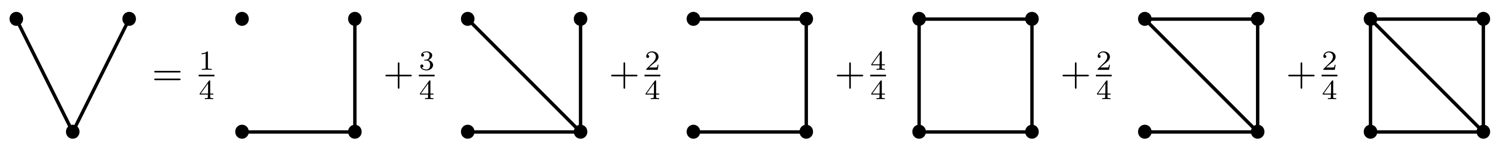
- defined $f_W(H) := d(H,W)$ and extended linearly
- aim: define multiplication on $\mathcal{A}$ preserved by f_W
  $f_W(H_1 \times H_2) = f_W(H_1) \cdot f_W(H_2)$
- $H_1 \times H_2 = \sum_H \frac{|\{(A,B) | V(H)=A \cup B, H[A] \cong H_1, H[B] \cong H_2\}|}{\binom{|H_1|+|H_2|}{|H_1|}} H$





Kernel of the map

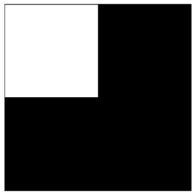
- defined $f_W(H) := d(H, W)$ and extended linearly
- Ker(f_W) always contains certain elements
  $f_W(K_2) = \frac{1}{3} f_W(\overline{K_{1,2}}) + \frac{2}{3} f_W(K_{1,2}) + \frac{3}{3} f_W(K_3)$



- let $\mathcal{A}'$ be the space generated by $H - \sum_{H'} d(H', H) H$
  $\mathcal{A}' \subseteq \mathrm{Ker}(f_W) \Rightarrow$ homomorphism $f_W : \mathcal{A}/\mathcal{A}' \to \mathbb{R}$

Rooted homomorphisms

- consider a graph G with a distinguish vertex (root)
  a random sample always includes the root
- algebra $\mathcal{A}^\bullet$ on combinations of rooted graphs
- rooted graphon $\rightarrow$ a homomorphism from $\mathcal{A}^\bullet$ to $\mathbb{R}$
  random choice of the root x_0 $\rightarrow$ probability distribution on homomorphisms f^{x_0} from $\mathcal{A}^\bullet$ to $\mathbb{R}$

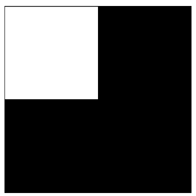


$$\frac{f^\bullet(K_2^\bullet) = 1/2,\ f^\bullet(\overline{K_2^\bullet}) = 1/2,\ f^\bullet(K_3^\bullet) = 1/4,\ \ldots}{f^\bullet(K_2^\bullet) = 1,\ f^\bullet(\overline{K_2^\bullet}) = 0,\ f^\bullet(K_3^\bullet) = 3/4,\ \ldots}$$

Rooted homomorphisms

- algebra $\mathcal{A}^\bullet$ of combinations of rooted graphs
  random choice of the root x_0 $\to$ probability distribution on homomorphisms f^{x_0} from $\mathcal{A}^\bullet$ to $\mathbb{R}$
- the value $f_W^{x_0}(H)$ for H with root v_0 is $\frac{k!}{|\mathrm{Aut}^\bullet(H)|}\times$

$$\int \prod_{v_i v_j \in E(H)} W(x_i,x_j) \prod_{v_i v_j \notin E(H)} (1-W(x_i,x_j))\,\mathrm{d}x_1\cdots x_k$$



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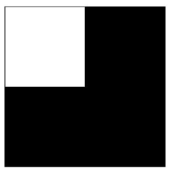
Questions?

General rooted graphs

- fix a graph R with vertices $r_1, \ldots, r_k$
  algebra $\mathcal{A}^R$ of combinations of R-rooted graphs
- random homomorphism f^R from $\mathcal{A}^R$ to $\mathbb{R}$
  random choice of the roots $x_1, \ldots, x_k$
  the roots do not induce $R \Rightarrow f^R \equiv 0$
  otherwise, sampling $|H| - k$ vertices $\Rightarrow$ prob. $f^R(H)$



$f^{K_2}(K_3^{K_2}) = 0,\ f^{K_2}(K_4^{K_2}) = 0,\ f^{K_2}(K_{1,2}^{K_2}) = 0,\ \ldots$

$f^{K_2}(K_3^{K_2}) = 1/2,\ f^{K_2}(K_4^{K_2}) = 1/4,\ f^{K_2}(K_{1,2}^{K_2}) = 1/2,\ \ldots$

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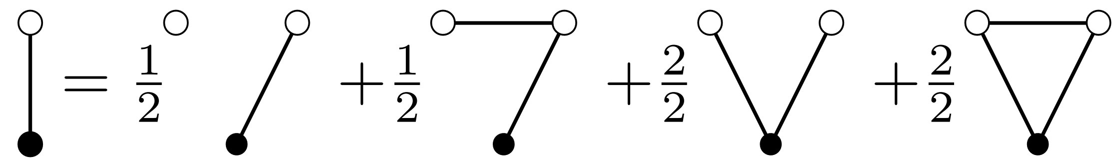
Operations with rooted graphs

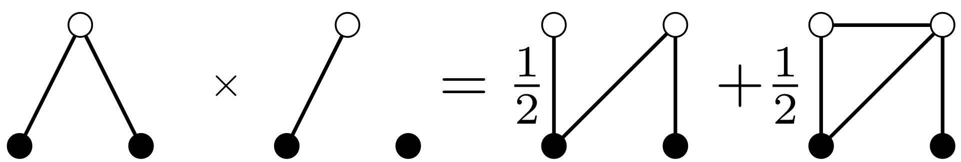
- projection

  prob. that deleting non-root vertices yields the flag



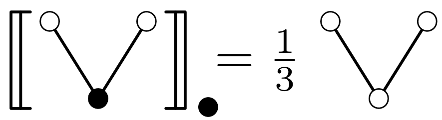
- multiplication

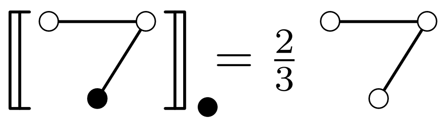
  prob. partitioning non-root vertices yields the terms



Expected value

- goal: $\mathbb{E}_R f_W^R(H) = f_W(\llbracket H \rrbracket_R)$ for $H \in \mathcal{A}^R$
- $f(\llbracket H \rrbracket_\bullet) = \mathbb{E}_z f^z(H)$





- $\llbracket \cdot \rrbracket_R : \mathcal{A}^R \to \mathcal{A} \qquad \llbracket H \rrbracket_R = \alpha H'$
  H' is the graph H without distinguishing roots
  α is the prob. that randomly chosen roots yield H

Questions?

Flag algebras

- algebra $\mathcal{A}$ of formal linear combinations of graphs
  addition and multiplication by a scalar

- homomorphism $f_W : \mathcal{A} \to \mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
  multiplication, elements always in $\mathrm{Ker}(f_W)$

- algebra $\mathcal{A}^R$ of R-rooted graphs
  random homomorphism $f_W^R : \mathcal{A}^R \to \mathbb{R}$
  multiplication, average operator $[\![\cdot]\!]_R : \mathcal{A}^R \to \mathcal{A}$
  $\mathbb{E}_R f_W^R(x) = f_W([\![x]\!]_R)$ for every $x \in \mathcal{A}^R$

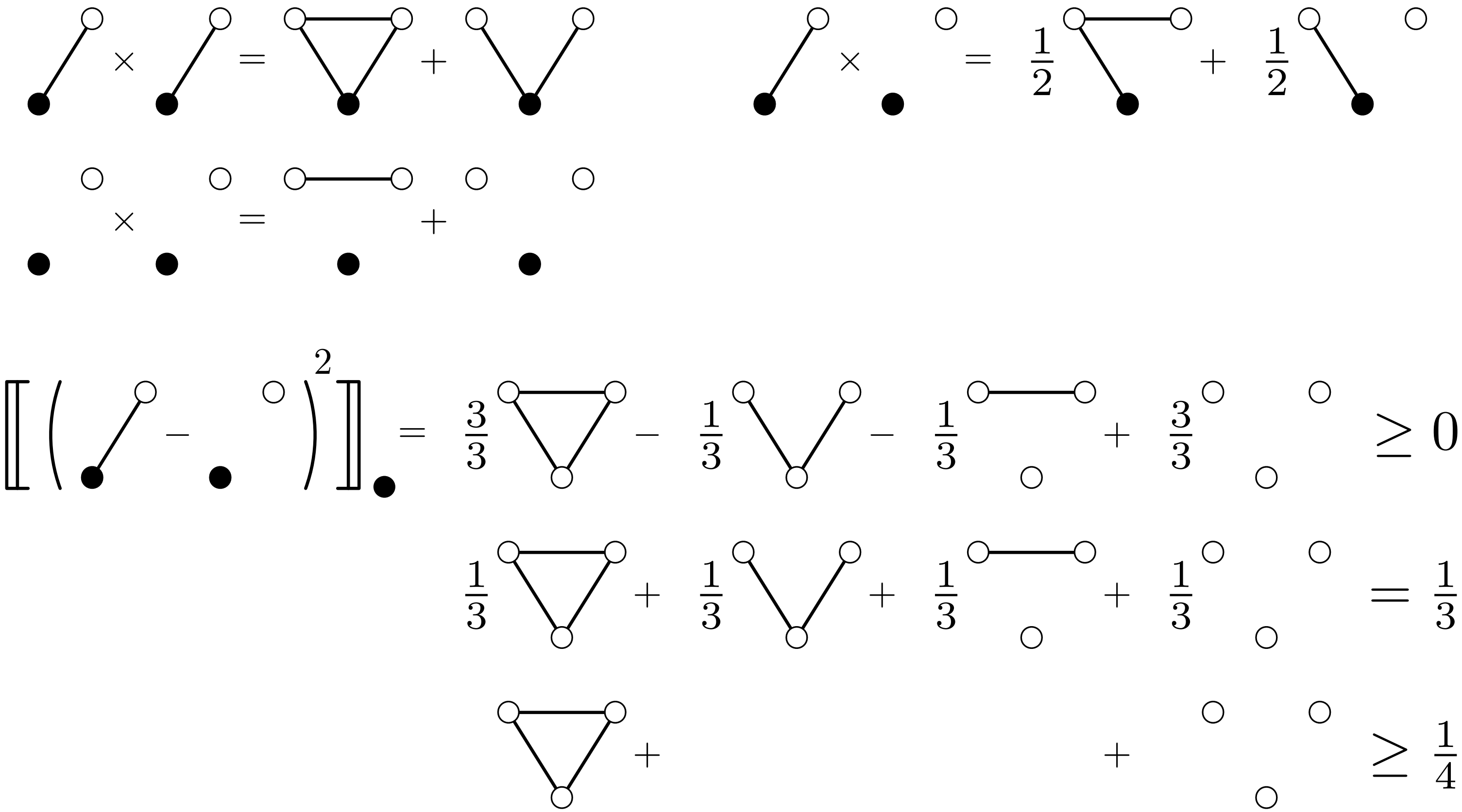
Computing with flags

- simple applications yields results such as
  $f_W(K_2) > 1/2 \Rightarrow f_W(K_3) > 0$ for every W
  $f_W(K_3 + \overline{K_3}) \geq 1/4$ for every W

- shorthand notation for $x, y \in \mathcal{A}$
  $x = y \Leftrightarrow \forall W \ f_W(x) = f_W(y)$
  $x \geq 0 \Leftrightarrow \forall W \ f_W(x) \geq 0$

- What can we use in computations?
  $x^2 \geq 0$ for every $x \in \mathcal{A}$
  $[\![x^2]\!]_R \geq 0$ for every $x \in \mathcal{A}^R$

Goodman's theorem



Questions?