Combinatorial limits

Part 1

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Warsaw May 2018

Graph limits

- large networks $\approx$ large graphs
  how to represent? how to model? how to generate?
- concise (analytic) representation of large graphs
  we implicitly use limits in our considerations anyway
- mathematics motivation – extremal graph theory
  What is a typical structure of an extremal graph?
  calculations avoiding smaller order terms
- today: dense graphs ($|E| = \Omega(|V|^2)$)
- convergence vs. analytic representation

Overview of the course

- Limits of dense graphs

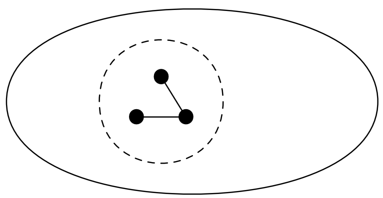
  Survey of main concepts in the area
- The flag algebra method

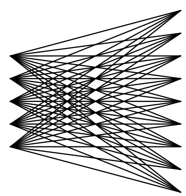
  Applications in extremal combinatorics
- Limits of sparse graphs

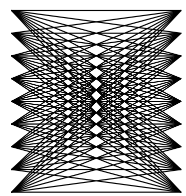
  Various concepts, less understood

Dense graph convergence

- convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
- extendable to other discrete structures





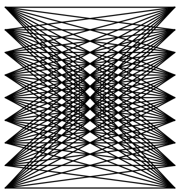


Convergent graph sequences

- complete graphs K_n
- complete bipartite graphs $K_{\alpha n,n}$
- Erdős-Rényi random graphs $G_{n,p}$
- any sequence of graphs with bounded maximum degree
- any sequence of planar graphs

Limit object: graphon

- graphon $W : [0,1]^2 \to [0,1]$
  measurable symmetric function, i.e. $W(x,y) = W(y,x)$
- "limit of adjacency matrices" (very imprecise)
- points of $[0,1] \approx$ vertices, values of $W \approx$ edge density



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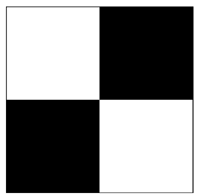

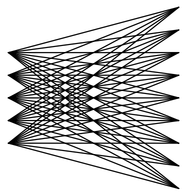

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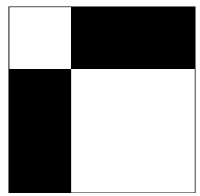
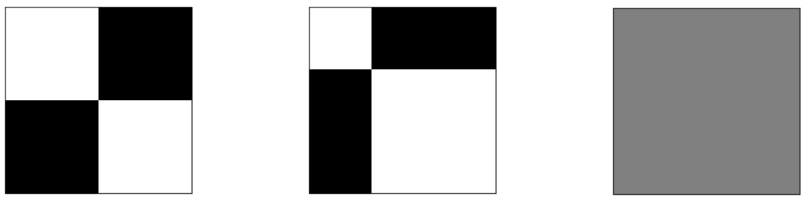


W-RANDOM GRAPHS

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  sample n random points $x_i \in [0,1] \approx$ vertices
  join two vertices by an edge with probability $W(x_i, x_j)$
- density of a graph H in a graphon W
  $d(H,W) =$ prob. $|H|$-vertex W-random graph is H



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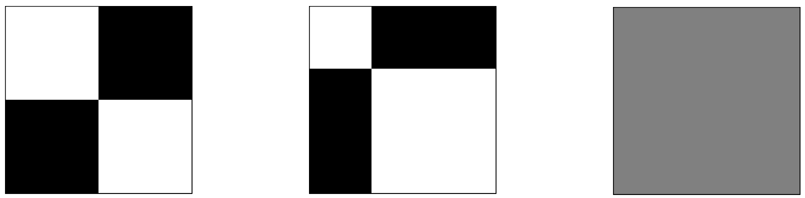
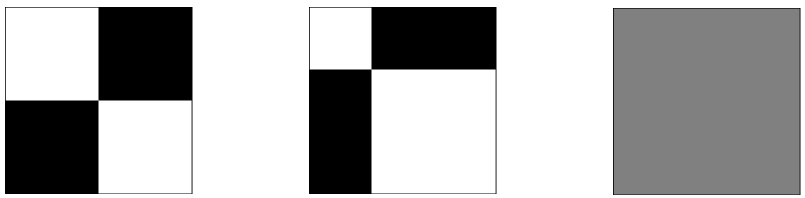
$$\frac{|H|!}{|\mathrm{Aut}(H)|} \int\limits_{[0,1]^{|H|}} \prod_{v_i v_j} W(x_i,x_j) \prod_{\overline{v_i v_j}} (1 - W(x_i,x_j)) \,\mathrm{d}x_1 \cdots x_n$$



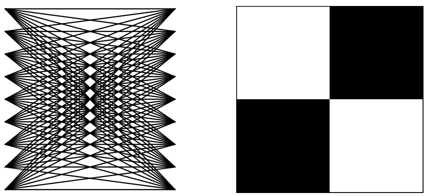
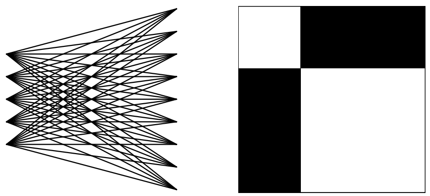
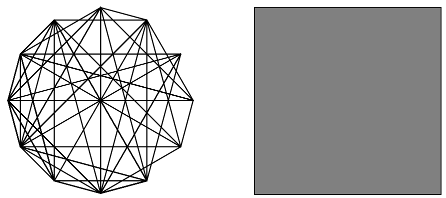
W-random graphs

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- $d(H,W)$ = expected density of H in a W-random graph
- $d(K_2,W) = \frac{1}{3}d(\overline{K_{1,2}},W) + \frac{2}{3}d(K_{1,2},W) + d(K_3,W)$
  Why? Integral. Random experiment.



W-random graphs

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- $d(H,W) =$ prob. $|H|$-vertex W-random graph is H
- $d(H,W) =$ expected density of H in a W-random graph
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$

Graphons as limits

- Does every convergent sequence have a limit?
- Uniqueness of a graphon representing a sequence.
- Is every graphon a limit of convergent sequence?

Martingales

- martingale is a sequence of random variables X_n
  $\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = X_n$ for every $n \in \mathbb{N}$
- Azuma-Hoeffding inequality
  suppose that $\mathbb{E}X_n = X_0$ and $|X_n - X_{n-1}| \leq c_n$
  $\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2e^{\frac{-t^2}{2\sum_{k=1}^{n} c_k^2}}$
- Doob's Martingale Convergence Theorem (corr.)
  if $|X_n| < K$, then $X_n \to X$ almost everywhere

W-random graphs converge

- A sequence of W-random graphs with increasing orders converges with probability one.
- fix $n \in \mathbb{N}$, a graph H and a graphon W
- X_i = exp. number of H in an n-vertex W-rand. graph after fixing the first i vertices and edges between them
- apply Azuma-Hoeffding inequality with $c_i = n^{|H|-1}$
  $\mathbb{P}\left(|X_n - X_0| \geq \varepsilon n^{|H|}\right) \leq 2e^{-\varepsilon^2 n/2}$

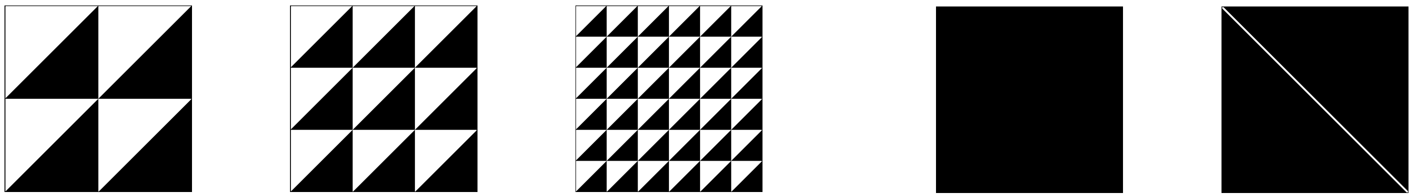
  $\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2e^{\frac{-t^2}{2\sum_{k=1}^{n} c_k^2}}$

W-RANDOM GRAPHS CONVERGE

- A sequence of W-random graphs with increasing orders converges with probability one.
- X_i = exp. number of H in an n-vertex W-rand. graph after fixing the first i vertices and edges between them $\mathbb{P}\left(\frac{|X_n - X_0|}{n^{|H|}} \geq \varepsilon\right) \leq 2e^{-\varepsilon^2 n/2}$
- the sum of $2e^{-\varepsilon^2 n/2}$ is finite for every $\varepsilon > 0$
- Borel-Cantelli $\Rightarrow$ the sequence converges with prob. one
- $X_0 \approx \frac{d(H,W)n^{|H|}}{|H|!}$ $\Rightarrow$ the graphon W is its limit

Uniqueness of the limit

- $W^{\varphi}(x,y) := W(\varphi(x),\varphi(y))$ for $\varphi : [0,1] \to [0,1]$
- $d(H,W) = d(H,W^{\varphi})$ if φ is measure preserving
- Theorem (Borgs, Chayes, Lovász)
  If $d(H,W_1) = d(H,W_2)$ for all graphs H,
  then there exist measure preserving maps φ_1 and φ_2
  such that $W_1^{\varphi_1} = W_2^{\varphi_2}$ almost everywhere.



Graph regularity

- Frieze-Kannan regularity, Szemerédi regularity
- $\forall \epsilon > 0 \; \exists K_\varepsilon$ such that every graph G has an ε-regular equipartition $V_1, \ldots, V_k$ with $k \leq K_\varepsilon$
  $||V_i| - |V_j|| \leq 1$ for all i and j
- equipartition $V_1, \ldots, V_k \to$ density matrix $A_{ij} = \frac{e(V_i, V_j)}{|V_i||V_j|}$
- $\forall \delta > 0, H \; \exists \varepsilon > 0$ such that the density matrix of an ε-regular partition determines $d(H, G)$ upto an δ-error
- the lemma holds with prepartitions

Existence of limit graphon

- fix a convergent sequence G_i, $i \in \mathbb{N}$, of graphs
- set $\varepsilon_j = 2^{-j}$ and fix ε_1-regular partition of G_i
  fix ε_{j+1}-regular partition refining the ε_j-regular one
- take a subsequence G'_i of G_i such that all but finitely many ε_j-regular partitions have the same num. parts
- let A^{ij} be the density matrix for G_i and ε_j
- take a subsequence G''_i of G'_i such that
  A^{ij} coordinate-wise converge for every j

Existence of limit graphon

- a convergent sequence G_i, density matrices A^{ij}
  let A^j be the coordinate-wise limit of A^{ij}
- interpret A^j as a random variable on $[0, 1]^2$ and
  apply Doob's Martingale Convergence Theorem
  in this way, we obtain a graphon W
- relate $d(H, W)$ to the density of H based on A^j

Questions?

Graphon entropy

- Hatami, Janson, Szegedy (2013)
  Falgas-Ravry, O'Connell, Strömberg, Uzzell

- How many graphs resemble a graphon W?
  the number $\approx 2^{cn^2/2+o(n^2)}$, what is c?
  $c = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{\log|n\text{-vertex graphs } \varepsilon\text{-close to } W|}{n^2/2}$

- graphon entropy $\mathrm{Ent}(W) = \int h(W(x,y))\mathrm{d}xy$
  where $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$

- the constant c is $\mathrm{Ent}(W)$

Questions?