Fact (Exercise)

If μ is a progress measures
then σ_μ is a winning strategy for Even

Hint Show that if a cycle has only progressive edges then it is even

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Exercise

- $|B_{d,k}| \leq \binom{k+\frac{d}{2}}{\frac{d}{2}}$
- If $k = n^{2/3}$ then $|B_{d,k}| = O(n^{d/3})$

Exercise If progress measure lifting is run with $B_{d,k}$ instead of $M_{d,n}$ then $\mu(v) \neq \infty$ for all $v \in V$ that are in any dominion of size $\leq k$

Exercise If $T(n,d) \leq n^{1/3} \cdot [T(n,d-1) + n^{d/3}]$
$T(n,2) \leq O(m \cdot n)$
then $T(n,d) = O(n^{d/3 + o(1)})$

Fact If $\mathcal{A}$ is a reachability (d,n)-separator and parity game G has $\leq n$ vertices and $\leq d$ priorities then games G and $\mathcal{A} \times G$ have the same winners

Proof Exercise; use positional determinacy

$\mathcal{C}_{d,n}$ is a reachability automaton:

- States $\{\langle c_d, c_{d-2}, \dots, c_2 \rangle : 0 \leq c_p \leq n\} \cup \{\text{accept}\}$
- Initial state: $(0,0,\dots,0)$
- $$\delta(\langle c_d, c_{d-2}, \dots, c_2 \rangle, p) = \begin{cases} \langle c_d, c_{d-2}, \dots, c_{p+1}, 0, \dots, 0 \rangle & \text{if } p \text{ odd} \\ \langle c_d, c_{d-2}, \dots, c_p + 1, 0, \dots, 0 \rangle & \text{if } p \text{ even and } c_p < n \\ \text{accept} & \text{if } p \text{ even and } c_p = n \end{cases}$$

Fact $\mathcal{C}_{d,n}$ is a reachability (d,n)-separator

Proof Exercise; prove that if $|G| \leq n$ then:

- $\mathcal{C}_{d,n}$ rejects all words in $\pi(\text{All Odd}(G))$
- $\mathcal{C}_{d,n}$ accepts all words in $\pi(\text{Even}(G)) \supseteq \pi(\text{AllEven}(G))$

$$S_{d,n} \stackrel{\text{df}}{=} \left\{ \langle s_{d-1}, s_{d-3}, \ldots, s_1 \rangle : s_p \in \{0,1\}^* \text{ and } \sum_{\text{odd } p} |s_p| \leq \lceil \lg n \rceil \right\}$$

__Exercise__

Prove that every ordered tree of height $\leq d/2$ and with $\leq n$ leaves has an isomorphic subtree in $S_{d,n}$

$S_{d,n}$ is a safety automaton:

- States $S_{d,n} \cup \{\text{reject}\}$
- Initial state: $\langle 11\ldots1, \varepsilon, \ldots, \varepsilon \rangle$
- $$\delta(s,p) = \begin{cases} \text{the lex-largest } s', \text{s.t. } s|_p >_{lex} s'|_p & \text{if } p \text{ odd} \\ \text{the lex-largest } s', \text{s.t. } s|_p \geq_{lex} s'|_p & \text{if } p \text{ even} \\ \text{reject} & \text{if none such } s' \text{ exists} \end{cases}$$

__Fact__ $S_{d,n}$ is a safety (d,n)-separator

__Proof__ __Exercise__; prove that if $|G| \leq n$ then:

- $S_{d,n}$ rejects all words in $\pi(\text{Odd}(G)) \supseteq \pi(\text{AllOdd}(G))$
- $S_{d,n}$ accepts all words in $\pi(\text{AllEven}(G))$