PhD Open lectures, University of Warsaw

Marcin Jurdziński (University of Warwick): Algorithms for solving parity games

Your solutions should be sent as a PDF to Paweł Parys (`parys@mimuw.edu.pl`) by **Monday, June 25, 2018**. To get a grade n, you need to solve $n-1$ problems.

A game graph is defined as a tuple:

- a graph in which every vertex has at least one successor,
- a partition of the set of its vertices into vertices belonging to player Even and vertices belonging to player Odd,
- a function assigning a priority to every vertex, and
- an initial vertex.

Problem 1. Given a game graph G we can consider two games:

a) a "standard" parity game: we consider infinite plays; the winner is determined by the highest priority seen infinitely often;

b) a "finite" parity game: we play until some vertex is visited twice; when the game ends in some vertex v, the play (the path in the graph) consists of a (possibly empty) path P from the initial vertex to v, and of a cycle from v to v; the winner is determined by the highest priority visited on the cycle C.

Prove that for every graph G, the "standard" parity game and the "finite" parity game have the same winner.

Problem 2. Suppose that a total deterministic reachability automaton $\mathscr{A}$ over alphabet $\{1, \ldots, d\}$ has the following properties:

1. If H is a graph with n vertices, labeled by priorities from $\{1, \ldots, d\}$, in which the maximal priority on every cycle is even, and if P is an infinite path in H, then $\mathscr{A}$ accepts $\pi(P)$.
2. If H is a graph with n vertices, labeled by priorities from $\{1, \ldots, d\}$, in which the maximal priority on every cycle is odd, and if P is an infinite path in H, then $\mathscr{A}$ rejects $\pi(P)$.

Prove that if G is a game graph with n vertices and priorities from $\{1, \ldots, d\}$, then the games G and $\mathscr{A} \times G$ have the same winner.

Remark. This fact strengthens the fact from the slides to the lecture (part 2, page 3, slide 6). The notation is the same as on the slides, in particular:

- $\pi(P)$ is the sequence of priorities seen on the infinite path P;
- the *synchronized product* $\mathscr{A} \times G$ is a reachability game, in which
  - the set of vertices is $Q \times V$, where Q is the set of states of $\mathscr{A}$, and V is the set of vertices of G;
  - there is an edge from (q, v) to (q', v'), if in G there is an edge from v to v' and in $\mathscr{A}$ there is a transition from q to q' reading $\pi(v)$ (where $\pi(v)$ is the priority assigned to v in G);
  - vertex (q, v) belongs to the player to which v belongs in G;
  - the initial vertex is (q_I, v_I), where q_I is the initial state of $\mathscr{A}$ and v_I is the initial vertex of G;
  - the objective for Even is to reach a pair (q, v) in which q is accepting.

Problem 3. The synchronized product can be also defined for nondeterministic automata. Let $\mathscr{A}$ be a total nondeterministic reachability automaton $\mathscr{A}$ over alphabet $\{1,\dots,d\}$, and let G be a game graph with priorities from $\{1,\dots,d\}$. Then the reachability game $\mathscr{A}\times G$ is defined as follows:

- the set of vertices is $Q\times V\times\{1,2\}$, where Q is the set of states of $\mathscr{A}$, and V is the set of vertices of G;
- there is an edge from $(q,v,1)$ to $(q',v,2)$, if in $\mathscr{A}$ there is a transition from q to q' reading $\pi(v)$ (where $\pi(v)$ is the priority assigned to v in G);
- there is an edge from $(q,v,2)$ to $(q,v',1)$, if in G there is an edge from v to v';
- vertices $(q,v,1)$ belong to Even, and a vertex $(q,v,2)$ belongs to the player to which v belongs in G (in other words: first Even chooses a transition of $\mathscr{A}$, and then an appropriate player moves in G);
- the initial vertex is $(q_I,v_I,1)$, where q_I is the initial state of $\mathscr{A}$ and v_I is the initial vertex of G;
- the objective for Even is to reach a pair $(q,v,2)$ in which q is accepting.

Prove that the fact from the slides to the lecture (part 2, page 3, slide 6) is false for nondeterministic automata: show a nondeterministic reachability (d,n)-separator $\mathscr{A}$ (for some d and n) and a game graph G with n vertices and priorities from $\{1,\dots,d\}$ such that the games G and $\mathscr{A}\times G$ do not have the same winner (a definition of a (d,n)-separator is given in the slides to the lecture).

Problem 4. Let $X_{d,n}$ be the set of all ordered trees that have n leaves on depth $d/2$ (and no leaves on other depths). Let $T_{d,n}$ be the ordered tree obtained by merging the roots of all trees from $X_{d,n}$ (i.e., for every $t\in X_{d,n}$, the children of the root of t are now children of the new root of $T_{d,n}$), where the order between nodes coming from different trees $t\in X_{d,n}$ is arbitrary but fixed.

Prove that if G is a game graph with n vertices and priorities in $\{1,\dots,d\}$ such that there is a progress measure for G (in some ordered tree T), then there is a progress measure for G in $T_{d,n}$ (this is an analogue of the fact from the slides to the lecture, part 2, page 2, slide 3, where the same is shown for a tree $M_{d,n}$).

Problem 5. We say that a game graph G with n vertices and priorities in $\{1,\dots,d\}$ is *nice* if the initial vertex belongs to a dominion of size at most $n^{1/\log(d)}$. Prove that there exists a polynomial algorithm solving nice parity games (i.e., a polynomial algorithm that given a nice game graph G says which player wins in G).

Remark. Recall from the lecture the definition of a *dominion*: a set of vertices D is a dominion for player X if X has s strategy that ensures winning from every vertex $v\in D$ without visiting any vertex not being in D.

Hint. It may be useful to analyze the slides to the lecture: part 2, page 3, slide 1 ("progress measures for small dominions"), and slides talking about succinct multi-counters (part 2, pages 4-5).