# Multi-join Query Evaluation on Big Data
# Lecture 1

Dan Suciu

March, 2015

# About Me

- Orignally from Romania
- Had fun with Math: 1976 IMO
- PhD from University of Pennsylvania: Parallel Query Languages
- Bell Labs and AT&T Labs: Semistructured Data, XML
- University of Washington: data privacy, probabilistic data, Big Data

Today's topic: Big Data!

# Course Organization

- Four lectures (1.5h): slides available on the course Website
- Two sections (1h): mostly interactive
- A problem set to pass the course: seven problems (simple to challenging); email me your solutions by April 30, 2015.
- I hope you can attend all lectures and sections: you need them in order to solve the problems.

# Multi-join Query Evaluation – Outline

Part 1 Optimal Sequential Algorithms. Thursday 14:15-15:45

Part 2 Lower bounds for Parallel Algorithms. Friday 14:15-15:45

Part 3 Optimal Parallel Algorithms. Saturday 9-10:30

Part 3 Data Skew. Saturday 11-12

# Multi-join Query Evaluation – Outline

Part 1 Optimal Sequential Algorithms. Thursday 14:15-15:45

Part 2 Lower bounds for Parallel Algorithms. Friday 14:15-15:45

Part 3 Optimal Parallel Algorithms. Saturday 9-10:30

Part 3 Data Skew. Saturday 11-12

# Multi-join Query Evaluation – Outline

Part 1 Optimal Sequential Algorithms. Thursday 14:15-15:45

Part 2 Lower bounds for Parallel Algorithms. Friday 14:15-15:45

Part 3 Optimal Parallel Algorithms. Saturday 9-10:30

Part 3 Data Skew. Saturday 11-12

# Multi-join Query Evaluation – Outline

Part 1 Optimal Sequential Algorithms. Thursday 14:15-15:45

Part 2 Lower bounds for Parallel Algorithms. Friday 14:15-15:45

Part 3 Optimal Parallel Algorithms. Saturday 9-10:30

Part 3 Data Skew. Saturday 11-12

# Bibliography


- E Friedgut, Hypergraphs, entropy, and inequalities, American Mathematical Monthly, 749-760, 2004.
- Albert Atserias, Martin Grohe, Dniel Marx: Size Bounds and Query Plans for Relational Joins. SIAM J. Comput. 42(4): 1737-1767 (2013)
- Hung Q. Ngo, Christopher Ré, Atri Rudra: Skew strikes back: new developments in the theory of join algorithms. SIGMOD Record 42(4): 5-16 (2013)
- Paul Beame, Paraschos Koutris, Dan Suciu: Skew in parallel query processing. PODS 2014: 212-223
- Paul Beame, Paraschos Koutris, Dan Suciu: Communication steps for parallel query processing. PODS 2013: 273-284

# Outline for Lecture 1

- Background: Queries, Databases, Query Evaluation
- The AGM inequality
- Friedgut's inequality
- Worst-case optimal query evaluation
- Summary

# Relations and Databases

Person

| Name | Age | City | Hobby |
|---|---|---|---|
| Alice | 22 | Lódtź | knitting |
| Bob | 33 | Lyon | karate |
| Carol | 44 | Lódtź | kayaking |
| David | 33 | Lima | karate |
| Eve | 22 | Lima | knitting |

**Schema** Relation/table name `Person`;
Attribute/column names `Name`, `Age`, `City`, `Hobby`;
Key `Name`

**Instance** Set of tuples/rows/records,
e.g. `(Alice, 22, Lódtź, knitting)`

**Size** Number of tuples $m = 5$; note: relation is a *set*

**Database** is a set of relations = a finite structure

# Relations and Databases

Person

| <u>Name</u> | Age | City | Hobby |
|---|---|---|---|
| Alice | 22 | Lódtź | knitting |
| Bob | 33 | Lyon | karate |
| Carol | 44 | Lódtź | kayaking |
| David | 33 | Lima | karate |
| Eve | 22 | Lima | knitting |

**Schema** Relation/table name `Person`;
Attribute/column names <u>`Name`</u>, `Age`, `City`, `Hobby`;
Key <u>`Name`</u>

**Instance** Set of tuples/rows/records,
e.g. `(Alice, 22, Lódtź, knitting)`

**Size** Number of tuples $m = 5$; note: relation is a *set*

**Database** is a set of relations = a finite structure

# Relations and Databases

Person

| Name | Age | City | Hobby |
|---|---|---|---|
| Alice | 22 | Lódtź | knitting |
| Bob | 33 | Lyon | karate |
| Carol | 44 | Lódtź | kayaking |
| David | 33 | Lima | karate |
| Eve | 22 | Lima | knitting |

**Schema** Relation/table name `Person`;
Attribute/column names `Name`, `Age`, `City`, `Hobby`;
Key `Name`

**Instance** Set of tuples/rows/records,
e.g. `(Alice, 22, Lódtź, knitting)`

**Size** Number of tuples $m = 5$; note: relation is a *set*

**Database** is a set of relations = a finite structure

# Relations and Databases

Person

| Name | Age | City | Hobby |
|---|---|---|---|
| Alice | 22 | Lódtź | knitting |
| Bob | 33 | Lyon | karate |
| Carol | 44 | Lódtź | kayaking |
| David | 33 | Lima | karate |
| Eve | 22 | Lima | knitting |

**Schema** Relation/table name `Person`;
Attribute/column names `Name`, `Age`, `City`, `Hobby`;
Key `Name`

**Instance** Set of tuples/rows/records,
e.g. `(Alice, 22, Lódtź, knitting)`

**Size** Number of tuples $m = 5$; note: relation is a *set*

**Database** is a set of relations = a finite structure

# Relations and Databases

Person

| Name | Age | City | Hobby |
|---|---|---|---|
| Alice | 22 | Lódtź | knitting |
| Bob | 33 | Lyon | karate |
| Carol | 44 | Lódtź | kayaking |
| David | 33 | Lima | karate |
| Eve | 22 | Lima | knitting |

**Schema** Relation/table name `Person`;
Attribute/column names `Name`, `Age`, `City`, `Hobby`;
Key `Name`

**Instance** Set of tuples/rows/records,
e.g. `(Alice, 22, Lódtź, knitting)`

**Size** Number of tuples $m = 5$; note: relation is a *set*

**Database** is a set of relations = a finite structure

# Basic Stuff that's Good To Know

- Relational database systems: Oracle, SQL Server, DB2, Postgres, SQLite, Dremel, Scope, Spark SQL
- Relations are flat (atomic values only): 1st normal form.
- Relations are persistent: stored in file systems, or in distributed file systems like Hadoop
- Physical data independence: system is allowed to organize the relation how it wishes. E.g. indexes, column-oriented DBs, partition on distributed servers, replicated.

# Relational Algebra

- Cartesian product / Join: $\bowtie$
- Projection: $\Pi_A$
- Selection: $\sigma_C$
- Union: $\cup$
- Set difference: $-$

This course: select-project-join

# Join

**$R \bowtie_{X=Y} S$**

The set of pairs $(t_1, t_2)$, with $t_1 \in R$ and $t_2 \in S$, s.t. $t_1.X = t_2.Y$

$R$

| $X$ | $U$ |
|---|---|
| $a_1$ | $b_1$ |
| $a_1$ | $b_2$ |
| $a_2$ | $b_3$ |
| $a_3$ | $b_4$ |

$S$

| $Y$ | $V$ |
|---|---|
| $a_1$ | $c_1$ |
| $a_1$ | $c_2$ |
| $a_3$ | $c_3$ |
| $a_4$ | $c_4$ |

$T = R \bowtie_{X=Y} S$

| $X$ | $U$ | $Y$ | $V$ |
|---|---|---|---|
| $a_1$ | $b_1$ | $a_1$ | $c_1$ |
| $a_1$ | $b_1$ | $a_1$ | $c_2$ |
| $a_1$ | $b_2$ | $a_1$ | $c_1$ |
| $a_1$ | $b_2$ | $a_1$ | $c_2$ |
| $a_3$ | $b_4$ | $a_3$ | $c_3$ |

Input schemas: $R(X,U)$, $S(Y,V)$
Output schema: $T(X,U,Y,V)$

# Natural Join

**$R \bowtie S$**

Joins $R, S$ on all common attributes, removes duplicate attributes

$R$

| $A$ | $B$ |
|---|---|
| $a_1$ | $b_1$ |
| $a_1$ | $b_2$ |
| $a_2$ | $b_3$ |
| $a_3$ | $b_4$ |

$S$

| $A$ | $C$ |
|---|---|
| $a_1$ | $c_1$ |
| $a_1$ | $c_2$ |
| $a_3$ | $c_3$ |
| $a_4$ | $c_4$ |

$T = R \bowtie S$

| $A$ | $B$ | $C$ |
|---|---|---|
| $a_1$ | $b_1$ | $c_1$ |
| $a_1$ | $b_1$ | $c_2$ |
| $a_1$ | $b_2$ | $c_1$ |
| $a_1$ | $b_2$ | $c_2$ |
| $a_3$ | $b_4$ | $c_3$ |

Input schemas: $R(A, B)$, $S(A, C)$
Output schema: $T(A, B, C)$

# Natural Join Examples

**Question**

In each case below: what is the output schema? What does the join do?

- $R(A, B, E, G) \bowtie S(A, C, D, E, F)$

# Natural Join Examples

**Question**

In each case below: what is the output schema? What does the join do?

- $R(A, B, E, G) \bowtie S(A, C, D, E, F)$
  Returns Output$(A, B, C, D, E, F, G) = R \bowtie_{(R.A=S.A) \wedge (R.E=S.E)} S$

# Natural Join Examples

**Question**

In each case below: what is the output schema? What does the join do?

- $R(A, B, E, G) \bowtie S(A, C, D, E, F)$
  Returns Output$(A, B, C, D, E, F, G) = R \bowtie_{(R.A=S.A)\wedge(R.E=S.E)} S$
- $R(A, B) \bowtie S(C, D, E)$

# Natural Join Examples

**Question**

In each case below: what is the output schema? What does the join do?

- $R(A, B, E, G) \bowtie S(A, C, D, E, F)$
  Returns Output$(A, B, C, D, E, F, G) = R \bowtie_{(R.A=S.A) \wedge (R.E=S.E)} S$
- $R(A, B) \bowtie S(C, D, E)$
  Returns the cartesian product: Output$(A, B, C, D, E) = R \times S$.

# Natural Join Examples

**Question**

In each case below: what is the output schema? What does the join do?

- $R(A, B, E, G) \bowtie S(A, C, D, E, F)$
  Returns $\text{Output}(A, B, C, D, E, F, G) = R \bowtie_{(R.A=S.A)\wedge(R.E=S.E)} S$
- $R(A, B) \bowtie S(C, D, E)$
  Returns the cartesian product: $\text{Output}(A, B, C, D, E) = R \times S$.
- $R(A, B) \bowtie S(A, B)$

# Natural Join Examples

**Question**

In each case below: what is the output schema? What does the join do?

- $R(A, B, E, G) \bowtie S(A, C, D, E, F)$
  Returns $\text{Output}(A, B, C, D, E, F, G) = R \bowtie_{(R.A=S.A) \wedge (R.E=S.E)} S$
- $R(A, B) \bowtie S(C, D, E)$
  Returns the cartesian product: $\text{Output}(A, B, C, D, E) = R \times S$.
- $R(A, B) \bowtie S(A, B)$
  Returns the intersection: $\text{Output}(A, B) = R \cap S$

# Very Quick Review of Basic Join Algorithms

Compute $R \bowtie_{A=B} S$

- Nested-loop join
- Hash-join
- Merge-join

(To describe in class.)

Complexity: $O((|R| + |S| + |R \bowtie_{A=B} S|) \log(|R| + |S|))$

Ignoring log factors, Complexity: $\boxed{O(|\text{Input}| + |\text{Output}|)}$

# Projection

**$\Pi_{AC}(T)$**

Projects $T$ on the attributes $A$ and $C$.

$T$

| $A$ | $B$ | $C$ |
|---|---|---|
| $a_1$ | $b_1$ | $c_1$ |
| $a_1$ | $b_1$ | $c_2$ |
| $a_1$ | $b_2$ | $c_1$ |
| $a_1$ | $b_2$ | $c_2$ |
| $a_3$ | $b_4$ | $c_3$ |

$\Pi_{AC}(T)$

| $A$ | $C$ |
|---|---|
| $a_1$ | $c_1$ |
| $a_1$ | $c_2$ |
| $a_3$ | $c_3$ |

Note: projection does duplicate elimination.

# Selection

$\sigma_{A=a}(R)$

Returns all rows where attribute $A$ has value $a$.

$R$

| $A$ | $B$ | $C$ |
|---|---|---|
| $a_1$ | $b_1$ | $c_1$ |
| $a_1$ | $b_1$ | $c_2$ |
| $a_1$ | $b_2$ | $c_1$ |
| $a_1$ | $b_2$ | $c_2$ |
| $a_3$ | $b_4$ | $c_3$ |

$\sigma_{C=c2}(R)$

| $A$ | $B$ | $C$ |
|---|---|---|
| $a_1$ | $b_1$ | $c_2$ |
| $a_1$ | $b_2$ | $c_2$ |

# Queries

## Relational Algebra

Defined alternatively as:

- Relational Algebra: $\{\bowtie, \sigma, \Pi, \cup, -\}$, or
- Relational Calculus, or First Order Logic: $\{\wedge, \vee, \exists, \forall, \neg, =\}$
- Non-recurisve datalog with negation, or
- A certain well-behaved fragment of SQL

## Conjunctive queries

Defined as:

- $\{\bowtie, \sigma, \Pi\}$, or
- $\{\wedge, \exists, =\}$, or
- A single datalog rule, or
- `select-from-where` SQL queries

This course: *full* conjunctive queries, meaning without $\Pi$.

# Conjunctive Queries

## Example

$Q_1(x, y, z, u) = R(x, y), S(y, z), T(z, u)$

- Relational Algebra: $(R(x, y) \bowtie S(y, z)) \bowtie T(z, u)$
- First Order Logic:
  $Q_1 = \{(x, y, z, u) \mid (x, y) \in R \wedge (y, z) \in S \wedge (z, u) \in T\}$

# Conjunctive Queries

## Example

$Q_1(x, y, z, u) = R(x, y), S(y, z), T(z, u)$

- Relational Algebra: $(R(x, y) \bowtie S(y, z)) \bowtie T(z, u)$
- First Order Logic:
  $Q_1 = \{(x, y, z, u) \mid (x, y) \in R \wedge (y, z) \in S \wedge (z, u) \in T\}$

## Example

$Q_2(x, u) = R(x, y), S(y, z), T(z, u)$

- Relational Algebra: $\Pi_{x,u}((R(x, y) \bowtie S(y, z)) \bowtie T(z, u))$
- First Order Logic:
  $Q_1 = \{(x, u) \mid \exists y \exists z((x, y) \in R \wedge (y, z) \in S \wedge (z, u) \in T)\}$

# Traditional Approach to Computing Conjunctive Queries

$$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$$

Optimizer generates a *query plan*:

$$\begin{aligned} \text{Temp}(x,y,z) =& R(x,y) \bowtie S(y,z) \\ Q(x,y,z) =& \text{Temp}(x,y,z) \bowtie T(z,x) \end{aligned}$$

Optimizers examines many possible plans, evaluates the cheapest plan.

Problem: intermediate results may be large, and very hard to estimate.

# Upper Bound on the Size of the Answer

Consider the join of two relations:

$$Q(x, y, z) = R(x, y), S(y, z)$$

**Question**

If $|R| = m_1, |S| = m_2$, how large can $|Q|$ be?

# Upper Bound on the Size of the Answer

Consider the join of two relations:

$$Q(x, y, z) = R(x, y), S(y, z)$$

**Question**

If $|R| = m_1, |S| = m_2$, how large can $|Q|$ be?

- Can be 0

# Upper Bound on the Size of the Answer

Consider the join of two relations:

$$Q(x, y, z) = R(x, y), S(y, z)$$

**Question**

If $|R| = m_1, |S| = m_2$, how large can $|Q|$ be?

- Can be 0
- Can be $m_1 m_2$

# Upper Bound on the Size of the Answer

Consider the join of two relations:

$$Q(x, y, z) = R(x, y), S(y, z)$$

**Question**

If $|R| = m_1, |S| = m_2$, how large can $|Q|$ be?

- Can be 0
- Can be $m_1 m_2$
- Answer: $0 \leq |Q| \leq m_1 m_2$.

# Upper Bound on the Size of the Answer

$$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$$

**Question**

If $|R| = m_1$, $|S| = m_2$, $|T| = m_3$, how large can the result be?

# Upper Bound on the Size of the Answer

$$Q(x, y, z) = R(x, y), S(y, z), T(z, x)$$

**Question**

If $|R| = m_1$, $|S| = m_2$, $|T| = m_3$, how large can the result be?

- Naive answer: $\leq m_1 m_2 m_3$ (why?)

# Upper Bound on the Size of the Answer

$$Q(x, y, z) = R(x, y), S(y, z), T(z, x)$$

**Question**

If $|R| = m_1$, $|S| = m_2$, $|T| = m_3$, how large can the result be?

- Naive answer: $\leq m_1 m_2 m_3$ (why?)
- Better answer: $\leq m_1 m_2$ (why?)

# Upper Bound on the Size of the Answer

$$Q(x, y, z) = R(x, y), S(y, z), T(z, x)$$

**Question**

If $|R| = m_1$, $|S| = m_2$, $|T| = m_3$, how large can the result be?

- Naive answer: $\leq m_1 m_2 m_3$ (why?)
- Better answer: $\leq m_1 m_2$ (why?)
- But also: $\leq m_1 m_3$, $\leq m_2 m_3$

# The Hypergraph of a Query

## Definition

Let $Q$ be a full conjunctive query without self-joins. The hypergraph $G$ of $Q$ consists of:

- Nodes($G$) = Vars($Q$) the set of variables of $Q$
- HyperEdges($G$) = Atoms($Q$) the set of atoms of $Q$.

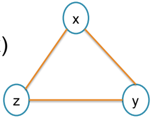
# The Hypergraph of a Query

## Definition

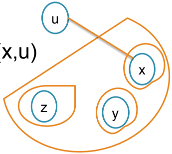
Let $Q$ be a full conjunctive query without self-joins. The hypergraph $G$ of $Q$ consists of:

- $\text{Nodes}(G) = \text{Vars}(Q)$ the set of variables of $Q$
- $\text{HyperEdges}(G) = \text{Atoms}(Q)$ the set of atoms of $Q$.

Q(x,y,z) = R(x,y),S(y,z),T(z,x)




Q(x,y,z) = R(x,y,z),S(x),T(y),K(z),M(x,u)



# Fractional Edge Cover / Vertex Packing of a Hypergraph $G$

$G$ = nodes $x_1, \ldots, x_k$ and hyperedges $R_1, \ldots, R_\ell$.

An *edge cover* = subset of edges that contain all nodes.

**Definition**

A *fractional edge cover* = sequence of positive numbers $u_1, \ldots, u_\ell$ s.t.:

$$\forall i : \sum_{j : x_i \in R_j} u_j \geq 1$$

Note: every edge cover is also a fractional edge cover (why?)

**Definition**

A *fractional vertex packing* = sequence of positive numbers $v_1, \ldots, v_k$ s.t.

$$\forall j : \sum_{i : x_i \in R_j} v_i \leq 1$$

Duality: $\min_{\mathbf{u}} \sum_j u_j = \max_{\mathbf{v}} \sum_i v_i = \rho^*$ = *fractional edge covering number*

# Fractional Edge Cover / Vertex Packing of a Hypergraph $G$

$G$ = nodes $x_1, \ldots, x_k$ and hyperedges $R_1, \ldots, R_\ell$.

An *edge cover* = subset of edges that contain all nodes.

**Definition**

A *fractional edge cover* = sequence of positive numbers $u_1, \ldots, u_\ell$ s.t.:

$$\forall i : \sum_{j : x_i \in R_j} u_j \geq 1$$

Note: every edge cover is also a fractional edge cover (why?)

**Definition**

A *fractional vertex packing* = sequence of positive numbers $v_1, \ldots, v_k$ s.t.

$$\forall j : \sum_{i : x_i \in R_j} v_i \leq 1$$

Duality: $\min_{\mathbf{u}} \sum_j u_j = \max_{\mathbf{v}} \sum_i v_i = \rho^*$ = *fractional edge covering number*

# Fractional Edge Cover / Vertex Packing of a Hypergraph $G$

$G$ = nodes $x_1, \ldots, x_k$ and hyperedges $R_1, \ldots, R_\ell$.

An *edge cover* = subset of edges that contain all nodes.

**Definition**

A *fractional edge cover* = sequence of positive numbers $u_1, \ldots, u_\ell$ s.t.:

$$\forall i : \sum_{j : x_i \in R_j} u_j \geq 1$$

Note: every edge cover is also a fractional edge cover (why?)

**Definition**

A *fractional vertex packing* = sequence of positive numbers $v_1, \ldots, v_k$ s.t.

$$\forall j : \sum_{i : x_i \in R_j} v_i \leq 1$$

Duality: $\min_{\mathbf{u}} \sum_j u_j = \max_{\mathbf{v}} \sum_i v_i = \rho^*$ = *fractional edge covering number*

# Fractional Edge Cover / Vertex Packing of a Hypergraph $G$

$G$ = nodes $x_1, \ldots, x_k$ and hyperedges $R_1, \ldots, R_\ell$.

An *edge cover* = subset of edges that contain all nodes.

## Definition

A *fractional edge cover* = sequence of positive numbers $u_1, \ldots, u_\ell$ s.t.:

$$\forall i : \sum_{j : x_i \in R_j} u_j \geq 1$$

Note: every edge cover is also a fractional edge cover (why?)

## Definition

A *fractional vertex packing* = sequence of positive numbers $v_1, \ldots, v_k$ s.t.

$$\forall j : \sum_{i : x_i \in R_j} v_i \leq 1$$

Duality: $\min_{\mathbf{u}} \sum_j u_j = \max_{\mathbf{v}} \sum_i v_i = \rho^*$ = *fractional edge covering number*

# Fractional Edge Cover / Vertex Packing of a Hypergraph $G$

$G$ = nodes $x_1, \ldots, x_k$ and hyperedges $R_1, \ldots, R_\ell$.

An *edge cover* = subset of edges that contain all nodes.

## Definition

A *fractional edge cover* = sequence of positive numbers $u_1, \ldots, u_\ell$ s.t.:

$$\forall i : \sum_{j: x_i \in R_j} u_j \geq 1$$

Note: every edge cover is also a fractional edge cover (why?)

## Definition

A *fractional vertex packing* = sequence of positive numbers $v_1, \ldots, v_k$ s.t.

$$\forall j : \sum_{i: x_i \in R_j} v_i \leq 1$$

Duality: $\min_{\mathbf{u}} \sum_j u_j = \max_{\mathbf{v}} \sum_i v_i = \rho^* =$ *fractional edge covering number*

# Fractional Edge Cover / Vertex Packing of a Hypergraph $G$

$G$ = nodes $x_1, \ldots, x_k$ and hyperedges $R_1, \ldots, R_\ell$.

An *edge cover* = subset of edges that contain all nodes.

## Definition

A *fractional edge cover* = sequence of positive numbers $u_1, \ldots, u_\ell$ s.t.:

$$\forall i : \sum_{j : x_i \in R_j} u_j \geq 1$$

Note: every edge cover is also a fractional edge cover (why?)

## Definition

A *fractional vertex packing* = sequence of positive numbers $v_1, \ldots, v_k$ s.t.

$$\forall j : \sum_{i : x_i \in R_j} v_i \leq 1$$

Duality: $\min_{\mathbf{u}} \sum_j u_j = \max_{\mathbf{v}} \sum_i v_i = \rho^*$ = *fractional edge covering number*

# AGM Inequality

Full conjunctive query: $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$
Relation sizes: $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$

**Proposition (Simple!)**

*Let $R_{i_1}, \ldots, R_{i_u}$ be any edge cover. Then* $\boxed{|Q| \leq m_{i_1} \cdot m_{i_2} \cdots m_{i_u}}$

(proof in class)

Atserias, Grohe and Marx proved:

**Theorem (AGM'13)**

*Let $u_1, \ldots, u_\ell$ be any fractional edge cover. Then* $\boxed{|Q| \leq m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}}$

We will prove it today. But first, let's see examples.

# AGM Inequality

Full conjunctive query: $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$
Relation sizes: $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$

## Proposition (Simple!)

*Let $R_{i_1}, \ldots, R_{i_u}$ be any edge cover. Then* $\boxed{|Q| \leq m_{i_1} \cdot m_{i_2} \cdots m_{i_u}}$

(proof in class)

Atserias, Grohe and Marx proved:

## Theorem (AGM'13)

*Let $u_1, \ldots, u_\ell$ be any fractional edge cover. Then* $\boxed{|Q| \leq m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}}$

We will prove it today. But first, let's see examples.

# AGM Inequality

Full conjunctive query: $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$
Relation sizes: $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$

**Proposition (Simple!)**

*Let $R_{i_1}, \ldots, R_{i_u}$ be any edge cover. Then* $\boxed{|Q| \leq m_{i_1} \cdot m_{i_2} \cdots m_{i_u}}$

(proof in class)

Atserias, Grohe and Marx proved:

**Theorem (AGM'13)**

*Let $u_1, \ldots, u_\ell$ be any fractional edge cover. Then* $\boxed{|Q| \leq m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}}$

We will prove it today. But first, let's see examples.

# AGM Inequality – A Simple Example

$$\boxed{AGM_{\mathbf{u}}(Q) = m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}}$$

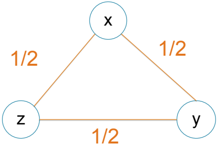
$Q(x, y, z) = R(x, y), S(y, z), T(z, x)$
$|R| = |S| = |T| = m$

# AGM Inequality – A Simple Example

$$\boxed{AGM_{\mathbf{u}}(Q) = m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}}$$

$Q(x, y, z) = R(x, y), S(y, z), T(z, x)$
$|R| = |S| = |T| = m$

A fractional edge: $\mathbf{u} = (1/2, 1/2, 1/2)$

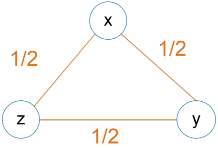


# AGM Inequality – A Simple Example

$$\boxed{AGM_{\mathbf{u}}(Q) = m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}}$$

$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$
$|R| = |S| = |T| = m$

A fractional edge: $\mathbf{u} = (1/2, 1/2, 1/2)$




It follows that $|Q| \leq m^{1/2} m^{1/2} m^{1/2} = m^{3/2}$

With $m$ (typed) edges you can built at most $m^{3/2}$ triangles!

# AGM Bound

## Definition

$AGM(Q) = \min_{\mathbf{u}} m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}$

Obviously: $|Q| \leq AGM(Q)$.

## Example

$Q(x,y,z) = R(x,y), S(y,z), T(z,x),$ $\quad |R| = m_1, |S| = m_2, |T| = m_3$

| $\mathbf{u} =$ | $(1,1,0)$ | $(1,0,1)$ | $(0,1,1)$ | $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ |
|---|---|---|---|---|
| $AGM(Q) =$ min of | $m_1 m_2$ | $m_1 m_3$ | $m_2 m_3$ | $(m_1 m_2 m_3)^{1/2}$ |

## Example

$Q(x,y,z,v,w) = R(x,y), S(y,z), T(z,v), K(v,w)$

| $\mathbf{u} =$ | $(1,0,1,1)$ | $(1,1,0,1)$ |
|---|---|---|
| $AGM(Q) =$ min of | $m_1 m_3 m_4$ | $m_1 m_2 m_4$ |

# AGM Bound

**Definition**

$AGM(Q) = \min_{\mathbf{u}} m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}$

Obviously: $|Q| \leq AGM(Q)$.

**Example**

$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$, $\quad |R| = m_1, |S| = m_2, |T| = m_3$

| | | | | |
|---|---|---|---|---|
| $\mathbf{u} =$ | $(1,1,0)$ | $(1,0,1)$ | $(0,1,1)$ | $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ |
| $AGM(Q) =$ min of | $m_1 m_2$ | $m_1 m_3$ | $m_2 m_3$ | $(m_1 m_2 m_3)^{1/2}$ |

**Example**

$Q(x,y,z,v,w) = R(x,y), S(y,z), T(z,v), K(v,w)$

| | | |
|---|---|---|
| $\mathbf{u} =$ | $(1,0,1,1)$ | $(1,1,0,1)$ |
| $AGM(Q) =$ min of | $m_1 m_3 m_4$ | $m_1 m_2 m_4$ |

# AGM Bound

## Definition

$AGM(Q) = \min_{\mathbf{u}} m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}$

Obviously: $|Q| \leq AGM(Q)$.

## Example

$Q(x, y, z) = R(x, y), S(y, z), T(z, x),$ $|R| = m_1, |S| = m_2, |T| = m_3$

| $\mathbf{u}$ = | $(1,1,0)$ | $(1,0,1)$ | $(0,1,1)$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ |
|---|---|---|---|---|
| $AGM(Q)$ = min of | $m_1 m_2$ | $m_1 m_3$ | $m_2 m_3$ | $(m_1 m_2 m_3)^{1/2}$ |

## Example

$Q(x, y, z, v, w) = R(x, y), S(y, z), T(z, v), K(v, w)$

| $\mathbf{u}$ = | $(1,0,1,1)$ | $(1,1,0,1)$ |
|---|---|---|
| $AGM(Q)$ = min of | $m_1 m_3 m_4$ | $m_1 m_2 m_4$ |

# AGM Bound v.s. Fractional Vertex Covering Number

$$\boxed{AGM_{\mathbf{u}}(Q) = m_1^{u_1} \cdot m_2^{u_2} \cdots m_\ell^{u_\ell}}$$

$AGM(Q) = \min_{\mathbf{u}} AGM_{\mathbf{u}}(Q)$ is the optimal solution to:

$$\text{minimize} \sum_j u_j \log m_j$$

$$\forall i : \sum_{j : x_i \in R_j} u_j \geq 1$$

Notice: when $m_1 = \cdots = m_\ell = m$ then $AGM(Q) = m^{\rho^*}$.

Next: we will prove the AGM bound

# Friedgut's Inequality

Cauchy-Schwartz: $\sum_i a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} (\sum_i b_i^2)^{\frac{1}{2}}$

# Friedgut's Inequality

Cauchy-Schwartz: $\sum_i a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} (\sum_i b_i^2)^{\frac{1}{2}}$

Triangle: $\sum_{i,j,k} a_{ij} b_{jk} c_{ki} \leq (\sum_{i,j} a_{ij}^2)^{\frac{1}{2}} (\sum_{j,k} b_{jk}^2)^{\frac{1}{2}} (\sum_{k,i} c_{ki}^2)^{\frac{1}{2}}$

# Friedgut's Inequality

Cauchy-Schwartz: $\sum_i a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} (\sum_i b_i^2)^{\frac{1}{2}}$

Triangle: $\sum_{i,j,k} a_{ij} b_{jk} c_{ki} \leq (\sum_{i,j} a_{ij}^2)^{\frac{1}{2}} (\sum_{j,k} b_{jk}^2)^{\frac{1}{2}} (\sum_{k,i} c_{ki}^2)^{\frac{1}{2}}$

Hölder $(u + v + w \geq 1)$: $\sum_i a_i b_i c_i \leq (\sum_i a_i^{\frac{1}{u}})^u (\sum_i b_i^{\frac{1}{v}})^v (\sum_i c_i^{\frac{1}{w}})^w$

# Friedgut's Inequality

Cauchy-Schwartz: $\sum_i a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} (\sum_i b_i^2)^{\frac{1}{2}}$

Triangle: $\sum_{i,j,k} a_{ij} b_{jk} c_{ki} \leq (\sum_{i,j} a_{ij}^2)^{\frac{1}{2}} (\sum_{j,k} b_{jk}^2)^{\frac{1}{2}} (\sum_{k,i} c_{ki}^2)^{\frac{1}{2}}$

Hölder ($u + v + w \geq 1$): $\sum_i a_i b_i c_i \leq (\sum_i a_i^{\frac{1}{u}})^u (\sum_i b_i^{\frac{1}{v}})^v (\sum_i c_i^{\frac{1}{w}})^w$

## Theorem (Friedgut'04)

*Let $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$ be a query and $u_1, \ldots, u_\ell$ be a fractional edge cover. Then:*

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}^{\frac{1}{u_1}}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}^{\frac{1}{u_\ell}}\right)^{u_\ell}}$$

What are the queries in the examples above?

# Friedgut's Inequality

Cauchy-Schwartz: $\sum_i a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} (\sum_i b_i^2)^{\frac{1}{2}}$

Triangle: $\sum_{i,j,k} a_{ij} b_{jk} c_{ki} \leq (\sum_{i,j} a_{ij}^2)^{\frac{1}{2}} (\sum_{j,k} b_{jk}^2)^{\frac{1}{2}} (\sum_{k,i} c_{ki}^2)^{\frac{1}{2}}$

Hölder ($u + v + w \geq 1$): $\sum_i a_i b_i c_i \leq (\sum_i a_i^{\frac{1}{u}})^u (\sum_i b_i^{\frac{1}{v}})^v (\sum_i c_i^{\frac{1}{w}})^w$

## Theorem (Friedgut'04)

*Let $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$ be a query and $u_1, \ldots, u_\ell$ be a fractional edge cover. Then:*

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell} \leq \left( \sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}^{\frac{1}{u_1}} \right)^{u_1} \cdots \left( \sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}^{\frac{1}{u_\ell}} \right)^{u_\ell}}$$

What are the queries in the examples above?

$Q_{\text{Cauchy-Schwartz}}(x) = R(x), S(x);$

# Friedgut's Inequality

Cauchy-Schwartz: $\sum_i a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} (\sum_i b_i^2)^{\frac{1}{2}}$

Triangle: $\sum_{i,j,k} a_{ij} b_{jk} c_{ki} \leq (\sum_{i,j} a_{ij}^2)^{\frac{1}{2}} (\sum_{j,k} b_{jk}^2)^{\frac{1}{2}} (\sum_{k,i} c_{ki}^2)^{\frac{1}{2}}$

Hölder ($u + v + w \geq 1$): $\sum_i a_i b_i c_i \leq (\sum_i a_i^{\frac{1}{u}})^u (\sum_i b_i^{\frac{1}{v}})^v (\sum_i c_i^{\frac{1}{w}})^w$

## Theorem (Friedgut'04)

*Let $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$ be a query and $u_1, \ldots, u_\ell$ be a fractional edge cover. Then:*

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}^{\frac{1}{u_1}}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}^{\frac{1}{u_\ell}}\right)^{u_\ell}}$$

What are the queries in the examples above?

$Q_{\text{Cauchy-Schwartz}}(x) = R(x), S(x);$

$Q_{\text{triangle}}(x, y, z) = R(x, y), S(y, z), T(z, x);$

# Friedgut's Inequality

Cauchy-Schwartz: $\sum_i a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} (\sum_i b_i^2)^{\frac{1}{2}}$

Triangle: $\sum_{i,j,k} a_{ij} b_{jk} c_{ki} \leq (\sum_{i,j} a_{ij}^2)^{\frac{1}{2}} (\sum_{j,k} b_{jk}^2)^{\frac{1}{2}} (\sum_{k,i} c_{ki}^2)^{\frac{1}{2}}$

Hölder ($u + v + w \geq 1$): $\sum_i a_i b_i c_i \leq (\sum_i a_i^{\frac{1}{u}})^u (\sum_i b_i^{\frac{1}{v}})^v (\sum_i c_i^{\frac{1}{w}})^w$

## Theorem (Friedgut'04)

*Let $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$ be a query and $u_1, \ldots, u_\ell$ be a fractional edge cover. Then:*

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}^{\frac{1}{u_1}}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}^{\frac{1}{u_\ell}}\right)^{u_\ell}}$$

What are the queries in the examples above?

$Q_{\text{Cauchy-Schwartz}}(x) = R(x), S(x)$;

$Q_{\text{triangle}}(x, y, z) = R(x, y), S(y, z), T(z, x)$;

$Q_{\text{Hölder}}(x) = R(x), S(x), T(x)$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \le \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

**Base Case.** $|\mathbf{x}| = 1$: $Q(x) = R_1(x), \ldots, R_\ell(x)$, $u_1 + \ldots + u_\ell \ge 1$

Prove: $\sum_x a_{1,x}^{u_1} \cdots a_{\ell,x}^{u_\ell} \le (\sum_x a_{1,x})^{u_1} \cdots (\sum_x a_{\ell,x})^{u_\ell}$ This is Hölder.

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

**Base Case.** $|\mathbf{x}| = 1$: $Q(x) = R_1(x), \ldots, R_\ell(x)$, $u_1 + \ldots + u_\ell \geq 1$

Prove: $\sum_x a_{1,x}^{u_1} \cdots a_{\ell,x}^{u_\ell} \leq (\sum_x a_{1,x})^{u_1} \cdots (\sum_x a_{\ell,x})^{u_\ell}$ This is Hölder.

**Induction Step.** Pick a variable $x$, and remove it. For example,

$Q(x, y, z) = R(x, y), S(y, z), T(z, x)$ becomes $Q'(y, z) = R'(y), S(y, z), T'(z)$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

**Base Case.** $|\mathbf{x}| = 1$: $Q(x) = R_1(x), \ldots, R_\ell(x)$, $u_1 + \ldots + u_\ell \geq 1$

Prove: $\sum_x a_{1,x}^{u_1} \cdots a_{\ell,x}^{u_\ell} \leq (\sum_x a_{1,x})^{u_1} \cdots (\sum_x a_{\ell,x})^{u_\ell}$ This is Hölder.

**Induction Step.** Pick a variable $x$, and remove it. For example,
$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$ becomes $Q'(y,z) = R'(y), S(y,z), T'(z)$

$$\sum_{xyz} a_{xy}^{u_1} b_{yz}^{u_2} c_{zx}^{u_3} = \sum_{yz} b_{yz}^{u_2} \sum_{x} a_{xy}^{u_1} c_{zx}^{u_3} \qquad \text{group by } \sum_x$$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

**Base Case.** $|\mathbf{x}| = 1$: $Q(x) = R_1(x), \ldots, R_\ell(x)$, $u_1 + \ldots + u_\ell \geq 1$

Prove: $\sum_x a_{1,x}^{u_1} \cdots a_{\ell,x}^{u_\ell} \leq (\sum_x a_{1,x})^{u_1} \cdots (\sum_x a_{\ell,x})^{u_\ell}$ This is Hölder.

**Induction Step.** Pick a variable $x$, and remove it. For example,

$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$ becomes $Q'(y,z) = R'(y), S(y,z), T'(z)$

$$\sum_{xyz} a_{xy}^{u_1} b_{yz}^{u_2} c_{zx}^{u_3} = \sum_{yz} b_{yz}^{u_2} \sum_x a_{xy}^{u_1} c_{zx}^{u_3} \qquad \text{group by } \textstyle\sum_x$$

$$\leq \sum_{yz} b_{yz}^{u_2} \Big(\sum_x a_{xy}\Big)^{u_1} \Big(\sum_x c_{zx}\Big)^{u_3} \qquad \text{Hölder } u_1 + u_3 \geq 1$$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

**Base Case.** $|\mathbf{x}| = 1$: $Q(x) = R_1(x), \ldots, R_\ell(x)$, $u_1 + \ldots + u_\ell \geq 1$

Prove: $\sum_x a_{1,x}^{u_1} \cdots a_{\ell,x}^{u_\ell} \leq (\sum_x a_{1,x})^{u_1} \cdots (\sum_x a_{\ell,x})^{u_\ell}$ This is Hölder.

**Induction Step.** Pick a variable $x$, and remove it. For example,
$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$ becomes $Q'(y,z) = R'(y), S(y,z), T'(z)$

$$\begin{aligned}
\sum_{xyz} a_{xy}^{u_1} b_{yz}^{u_2} c_{zx}^{u_3} &= \sum_{yz} b_{yz}^{u_2} \sum_x a_{xy}^{u_1} c_{zx}^{u_3} && \text{group by } \textstyle\sum_x \\
&\leq \sum_{yz} b_{yz}^{u_2} \Big(\sum_x a_{xy}\Big)^{u_1} \Big(\sum_x c_{zx}\Big)^{u_3} && \text{Hölder } u_1 + u_3 \geq 1 \\
&= \sum_{yz} b_{yz}^{u_2} A_y^{u_1} C_z^{u_3} \leq \Big(\sum_{yz} b_{yz}\Big)^{u_2} \Big(\sum_y A_y\Big)^{u_1} \Big(\sum_z C_z\Big)^{u_3} && \text{Induction for } Q'
\end{aligned}$$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\textstyle\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

**Base Case.** $|\mathbf{x}| = 1$: $Q(x) = R_1(x), \ldots, R_\ell(x)$, $u_1 + \ldots + u_\ell \geq 1$

Prove: $\sum_x a_{1,x}^{u_1} \cdots a_{\ell,x}^{u_\ell} \leq (\sum_x a_{1,x})^{u_1} \cdots (\sum_x a_{\ell,x})^{u_\ell}$ This is Hölder.

**Induction Step.** Pick a variable $x$, and remove it. For example,
$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$ becomes $Q'(y,z) = R'(y), S(y,z), T'(z)$

$$\begin{aligned}
\sum_{xyz} a_{xy}^{u_1} b_{yz}^{u_2} c_{zx}^{u_3} &= \sum_{yz} b_{yz}^{u_2} \sum_{x} a_{xy}^{u_1} c_{zx}^{u_3} && \text{group by } \textstyle\sum_x \\
&\leq \sum_{yz} b_{yz}^{u_2} \Big(\sum_x a_{xy}\Big)^{u_1} \Big(\sum_x c_{zx}\Big)^{u_3} && \text{Hölder } u_1 + u_3 \geq 1 \\
&= \sum_{yz} b_{yz}^{u_2} A_y^{u_1} C_z^{u_3} \leq \Big(\sum_{yz} b_{yz}\Big)^{u_2} \Big(\sum_y A_y\Big)^{u_1} \Big(\sum_z C_z\Big)^{u_3} && \text{Induction for } Q' \\
&= \Big(\sum_{yz} b_{yz}\Big)^{u_2} \Big(\sum_{xy} a_{xy}\Big)^{u_1} \Big(\sum_{zx} c_{zx}\Big)^{u_3}
\end{aligned}$$

# Friedgut's Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

$$\boxed{\sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell}}$$

**Proof:** by induction on $|\mathbf{x}|$

**Base Case.** $|\mathbf{x}| = 1$: $Q(x) = R_1(x), \ldots, R_\ell(x)$, $u_1 + \ldots + u_\ell \geq 1$

Prove: $\sum_x a_{1,x}^{u_1} \cdots a_{\ell,x}^{u_\ell} \leq (\sum_x a_{1,x})^{u_1} \cdots (\sum_x a_{\ell,x})^{u_\ell}$ This is Hölder.

**Induction Step.** Pick a variable $x$, and remove it. For example,

$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$ becomes $Q'(y,z) = R'(y), S(y,z), T'(z)$

$$\sum_{xyz} a_{xy}^{u_1} b_{yz}^{u_2} c_{zx}^{u_3} = \sum_{yz} b_{yz}^{u_2} \sum_x a_{xy}^{u_1} c_{zx}^{u_3} \qquad \text{group by } \textstyle\sum_x$$

$$\leq \sum_{yz} b_{yz}^{u_2} \Big(\sum_x a_{xy}\Big)^{u_1} \Big(\sum_x c_{zx}\Big)^{u_3} \qquad \text{Hölder } u_1 + u_3 \geq 1$$

$$= \sum_{yz} b_{yz}^{u_2} A_y^{u_1} C_z^{u_3} \leq \Big(\sum_{yz} b_{yz}\Big)^{u_2} \Big(\sum_y A_y\Big)^{u_1} \Big(\sum_z C_z\Big)^{u_3} \qquad \text{Induction for } Q'$$

$$= \Big(\sum_{yz} b_{yz}\Big)^{u_2} \Big(\sum_{xy} a_{xy}\Big)^{u_1} \Big(\sum_{zx} c_{zx}\Big)^{u_3}$$

**QED**

# The AGM Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \dots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \dots, u_\ell$

Sizes $|R_1| = m_1, \dots, |R_\ell| = m_\ell$

Prove $\boxed{|Q| \leq m_1^{u_1} \cdots m_\ell^{u_\ell}}$

Let Dom = the domain of all constants in the relations $R_1, \dots, R_\ell$.

For every $j = 1, \dots, \ell$, and every tuple $\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}$, define:

$$a_{j,\mathbf{x}_j} = \begin{cases} 1 & \text{if the tuple } \mathbf{x}_j \text{ belongs to } R_j \\ 0 & \text{otherwise} \end{cases}$$

Then: $m_j = |R_j| = \sum_{\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}} a_{j,\mathbf{x}_j}$, $|Q| = \sum_{\mathbf{x} \in \mathrm{Dom}^{|\mathbf{x}|}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell}$

Now use Friedgut's inequality:

$$|Q| = \sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \Big(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\Big)^{u_1} \cdots \Big(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\Big)^{u_\ell} = m_1^{u_1} \cdots m_\ell^{u_\ell}$$

**QED**

# The AGM Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \dots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \dots, u_\ell$

Sizes $|R_1| = m_1, \dots, |R_\ell| = m_\ell$

Prove $\boxed{|Q| \leq m_1^{u_1} \cdots m_\ell^{u_\ell}}$

Let Dom = the domain of all constants in the relations $R_1, \dots, R_\ell$.

For every $j = 1, \dots, \ell$, and every tuple $\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}$, define:

$$a_{j,\mathbf{x}_j} = \begin{cases} 1 & \text{if the tuple } \mathbf{x}_j \text{ belongs to } R_j \\ 0 & \text{otherwise} \end{cases}$$

Then: $m_j = |R_j| = \sum_{\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}} a_{j,\mathbf{x}_j}$, $|Q| = \sum_{\mathbf{x} \in \mathrm{Dom}^{|\mathbf{x}|}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell}$

Now use Friedgut's inequality:

$$|Q| = \sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \Big(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\Big)^{u_1} \cdots \Big(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\Big)^{u_\ell} = m_1^{u_1} \cdots m_\ell^{u_\ell}$$

**QED**

# The AGM Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

Sizes $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$

Prove $\boxed{|Q| \leq m_1^{u_1} \cdots m_\ell^{u_\ell}}$

Let Dom = the domain of all constants in the relations $R_1, \ldots, R_\ell$.

For every $j = 1, \ldots, \ell$, and every tuple $\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}$, define:

$$a_{j,\mathbf{x}_j} = \begin{cases} 1 & \text{if the tuple } \mathbf{x}_j \text{ belongs to } R_j \\ 0 & \text{otherwise} \end{cases}$$

Then: $m_j = |R_j| = \sum_{\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}} a_{j,\mathbf{x}_j}$, $|Q| = \sum_{\mathbf{x} \in \mathrm{Dom}^{|\mathbf{x}|}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell}$

Now use Friedgut's inequality:

$$|Q| = \sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \Big(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\Big)^{u_1} \cdots \Big(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\Big)^{u_\ell} = m_1^{u_1} \cdots m_\ell^{u_\ell}$$

QED

# The AGM Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$
Sizes $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$
Prove $\boxed{|Q| \le m_1^{u_1} \cdots m_\ell^{u_\ell}}$
Let Dom = the domain of all constants in the relations $R_1, \ldots, R_\ell$.
For every $j = 1, \ldots, \ell$, and every tuple $\mathbf{x}_j \in \text{Dom}^{|\mathbf{x}_j|}$, define:

$$a_{j,\mathbf{x}_j} = \begin{cases} 1 & \text{if the tuple } \mathbf{x}_j \text{ belongs to } R_j \\ 0 & \text{otherwise} \end{cases}$$

Then: $m_j = |R_j| = \sum_{\mathbf{x}_j \in \text{Dom}^{|\mathbf{x}_j|}} a_{j,\mathbf{x}_j}$, $|Q| = \sum_{\mathbf{x} \in \text{Dom}^{|\mathbf{x}|}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell}$
Now use Friedgut's inequality:

$$|Q| = \sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \le \Big(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\Big)^{u_1} \cdots \Big(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\Big)^{u_\ell} = m_1^{u_1} \cdots m_\ell^{u_\ell}$$

QED

# The AGM Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$
Sizes $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$
Prove $\boxed{|Q| \leq m_1^{u_1} \cdots m_\ell^{u_\ell}}$
Let Dom = the domain of all constants in the relations $R_1, \ldots, R_\ell$.
For every $j = 1, \ldots, \ell$, and every tuple $\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}$, define:

$$a_{j,\mathbf{x}_j} = \begin{cases} 1 & \text{if the tuple } \mathbf{x}_j \text{ belongs to } R_j \\ 0 & \text{otherwise} \end{cases}$$

Then: $m_j = |R_j| = \sum_{\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}} a_{j,\mathbf{x}_j}$, $|Q| = \sum_{\mathbf{x} \in \mathrm{Dom}^{|\mathbf{x}|}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell}$
Now use Friedgut's inequality:

$$|Q| = \sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \Big(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\Big)^{u_1} \cdots \Big(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\Big)^{u_\ell} = m_1^{u_1} \cdots m_\ell^{u_\ell}$$

QED

# The AGM Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$

Sizes $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$

Prove $\boxed{|Q| \leq m_1^{u_1} \cdots m_\ell^{u_\ell}}$

Let Dom = the domain of all constants in the relations $R_1, \ldots, R_\ell$.

For every $j = 1, \ldots, \ell$, and every tuple $\mathbf{x}_j \in \text{Dom}^{|\mathbf{x}_j|}$, define:

$$a_{j,\mathbf{x}_j} = \begin{cases} 1 & \text{if the tuple } \mathbf{x}_j \text{ belongs to } R_j \\ 0 & \text{otherwise} \end{cases}$$

Then: $m_j = |R_j| = \sum_{\mathbf{x}_j \in \text{Dom}^{|\mathbf{x}_j|}} a_{j,\mathbf{x}_j}$, $|Q| = \sum_{\mathbf{x} \in \text{Dom}^{|\mathbf{x}|}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell}$

Now use Friedgut's inequality:

$$|Q| = \sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \left(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\right)^{u_1} \cdots \left(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\right)^{u_\ell} = m_1^{u_1} \cdots m_\ell^{u_\ell}$$

QED

# The AGM Inequality – Proof

Query $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$, fractional cover $u_1, \ldots, u_\ell$
Sizes $|R_1| = m_1, \ldots, |R_\ell| = m_\ell$
Prove $\boxed{|Q| \leq m_1^{u_1} \cdots m_\ell^{u_\ell}}$
Let Dom = the domain of all constants in the relations $R_1, \ldots, R_\ell$.
For every $j = 1, \ldots, \ell$, and every tuple $\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}$, define:

$$a_{j,\mathbf{x}_j} = \begin{cases} 1 & \text{if the tuple } \mathbf{x}_j \text{ belongs to } R_j \\ 0 & \text{otherwise} \end{cases}$$

Then: $m_j = |R_j| = \sum_{\mathbf{x}_j \in \mathrm{Dom}^{|\mathbf{x}_j|}} a_{j,\mathbf{x}_j}$, $|Q| = \sum_{\mathbf{x} \in \mathrm{Dom}^{|\mathbf{x}|}} a_{1,\mathbf{x}_1} \cdots a_{\ell,\mathbf{x}_\ell}$
Now use Friedgut's inequality:

$$|Q| = \sum_{\mathbf{x}} a_{1,\mathbf{x}_1}^{u_1} \cdots a_{\ell,\mathbf{x}_\ell}^{u_\ell} \leq \Big(\sum_{\mathbf{x}_1} a_{1,\mathbf{x}_1}\Big)^{u_1} \cdots \Big(\sum_{\mathbf{x}_\ell} a_{\ell,\mathbf{x}_\ell}\Big)^{u_\ell} = m_1^{u_1} \cdots m_\ell^{u_\ell}$$

**QED**

# Computing Full Conjunctive Queries

- Recall: all database systems compute one join at a time
- This may be much larger than the maximum output size, $AGM(Q)$.
- Goal: design a algorithm that runs in time $AGM(Q)$.

*Worst-Case-Optimal* algorithm: runs in time $AGM(Q)$.

# Worst-Case Optimal Algorithm

History:

- An algorithm that runs in time $O(n \cdot AGM(Q))$ was given in [AGM'2013].
- First worst-case optimal algorithm that was published: the NPRR algorithm by Ngo, Porat, Ré, Rudra, in PODS'2012. It is complex.
- Earlier algorithm *Leapfrog Trie-join* (LFTJ), by LogicBlox. Veldhuizen proved in ICDT'2014 that LFTJ is also worst case optimal.
- Ngo, Ré, Rudra gave a very simple worst-case algorithm, with a very simple optimality proof, in SIGMOD Records'2013. The algorithm is called *Generic Join*.

Next: we discuss Generic-Join

# Generic Join

**Compute $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$**

If $|\mathbf{x}| = 1$ then return $R_1 \cap \cdots \cap R_\ell$.

Otherwise, choose a variable $x$, occurring in atoms $R_{i_1}, \ldots, R_{i_k}$

- Compute $A = \Pi_x(R_{i_1}) \cap \cdots \cap \Pi_x(R_{i_k})$
- For each $a \in A$, compute $\texttt{Result}_a = Q[a/x]$ using *Generic-Join*
- Return $\bigcup_a \texttt{Result}_a$.

Runtime: $O(AGM(Q))$ (Recall: we ignore log-factors)

**$Q(x, y, z) = R(x, y), S(y, z), T(z, x)$**

- Compute $A = \Pi_x(R) \cap \Pi_x(T) = \{a_1, \ldots, a_n\}$
- For each $a_i \in A$, denote $R'(y) = R(a_i, y), T'(z) = T(z, a_i)$
  Compute $\texttt{Result}_i(a_i, y, z) = R'(y), S(y, z), T'(z)$
- Return $\bigcup_i \texttt{Result}_i$

Runtime: $O(m^{3/2})$ assuming $|R| = |S| = |T| = m$.

# Generic Join

## Compute $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$

If $|\mathbf{x}| = 1$ then return $R_1 \cap \cdots \cap R_\ell$.
Otherwise, choose a variable $x$, occurring in atoms $R_{i_1}, \ldots, R_{i_k}$

- Compute $A = \Pi_x(R_{i_1}) \cap \cdots \cap \Pi_x(R_{i_k})$
- For each $a \in A$, compute $\texttt{Result}_a = Q[a/x]$ using *Generic-Join*
- Return $\bigcup_a \texttt{Result}_a$.

Runtime: $O(AGM(Q))$ (Recall: we ignore log-factors)

## $Q(x, y, z) = R(x, y), S(y, z), T(z, x)$

- Compute $A = \Pi_x(R) \cap \Pi_x(T) = \{a_1, \ldots, a_n\}$
- For each $a_i \in A$, denote $R'(y) = R(a_i, y), T'(z) = T(z, a_i)$
  Compute $\texttt{Result}_i(a_i, y, z) = R'(y), S(y, z), T'(z)$
- Return $\bigcup_i \texttt{Result}_i$

Runtime: $O(m^{3/2})$ assuming $|R| = |S| = |T| = m$.

# Generic Join

**Compute $Q(\mathbf{x}) = R_1(\mathbf{x}_1), \ldots, R_\ell(\mathbf{x}_\ell)$**

If $|\mathbf{x}| = 1$ then return $R_1 \cap \cdots \cap R_\ell$.
Otherwise, choose a variable $x$, occurring in atoms $R_{i_1}, \ldots, R_{i_k}$

- Compute $A = \Pi_x(R_{i_1}) \cap \cdots \cap \Pi_x(R_{i_k})$
- For each $a \in A$, compute $\texttt{Result}_a = Q[a/x]$ using *Generic-Join*
- Return $\bigcup_a \texttt{Result}_a$.

Runtime: $O(AGM(Q))$ (Recall: we ignore log-factors)

**$Q(x,y,z) = R(x,y), S(y,z), T(z,x)$**

- Compute $A = \Pi_x(R) \cap \Pi_x(T) = \{a_1, \ldots, a_n\}$
- For each $a_i \in A$, denote $R'(y) = R(a_i, y)$, $T'(z) = T(z, a_i)$
  Compute $\texttt{Result}_i(a_i, y, z) = R'(y), S(y,z), T'(z)$
- Return $\bigcup_i \texttt{Result}_i$

Runtime: $O(m^{3/2})$ assuming $|R| = |S| = |T| = m$.

# Discussion: Generic Join v.s. Yannakakis' Alogirithm

[Yannakakis'82] described an algorithm for computing any *acyclic query* in time $O(|\text{Input}| + |\text{Output}|)$. Basic idea: first perform a *semijoin reduction* to ensure that all intermediate results are $\leq |\text{Output}|$, then compute the query in standard fashion, one join at a time.

$Q(x_0, x_1, x_2, x_3, x_4, x_5) =$
$R_1(x_0, x_1), R_2(x_1, x_2), R_3(x_2, x_3), R_4(x_3, x_4), R_5(x_4, x_5)$

$|R_1| = \ldots = |R_5| = m$, $AGM(Q) = m^3$ (optimal cover: $(1, 0, 1, 0, 1)$).

There are instances where $Q = \emptyset$, hence Yannakakis' algorithm takes time $O(m)$, yet Generic-join takes time $\Omega(m^3)$ (Discuss in class).

Newer work on *instance-optimal join algorithms* [Ngo'2014]

# Summary of Lecture 1

- Joins, and conjunctive queries are very important: in SQL, in data analytics, everywhere
- All traditional query processing algorithms compute one join at a time (except LogicBlox!): suboptimal.
- The AGM bound gives a tight upper bound on the query size, expressed in terms of *fractional edge cover*.
- The Generic-Join algorithm computes the query in time bounded by the AGM bound: hence *worst-case optimal*.