A Toolbox for Online Algorithms
The Exercises

Problem 1. In the fractional variant of the ski rental problem, at each day the algorithm decides what fraction $x \in [0,1]$ of the skis to buy. (The value of x cannot decrease, i.e., the algorithm can only buy an additional fraction of skis, and cannot sell them.) The remaining part of the skis have to be rented.

The following linear program (P_k) describes this optimization problem after k days, where z_j is the part of skis which have to be rented at day j.

$$\begin{aligned}
\text{minimize: } & B \cdot x + \sum_{j=1}^{k} z_j \\
\text{subject to: } & x + z_j \geq 1 && \text{for each day } j \leq k \\
& x \geq 0, \;\; z_j \geq 0
\end{aligned}$$

Note that the dual program (D_k) is the following:

$$\begin{aligned}
\text{maximize: } & \sum_{j=1}^{k} y_j \\
\text{subject to: } & \sum_{j=1}^{k} y_j \leq B \\
& 0 \leq y_j \leq 1 && \text{for each day } j \leq k
\end{aligned}$$

Initially, we set $x = 0$. Consider the following scheme for updating the variables at day k:

1. $z_k \leftarrow 1 - x$

2. $x \leftarrow (1 + a) \cdot x + c$

3. $y_k \leftarrow 1$

where a and c are some constants which have to be defined later. Use the linear programming method to show that this algorithm is R-competitive where

$$R = \frac{(1 + 1/B)^B}{(1 + 1/B)^B - 1} \quad .$$

Hint: choose a, so that the ΔP_i does not depend on x.

Problem 2. You have 1 Euro and your goal is to exchange it to polish zlotys during the next k consecutive days. The exchange rate is an arbitrary function from days to real numbers from the interval $[1, M]$, where M is known to the algorithm.

More precisely, at day $i \in \{1, \ldots, k\}$, you learn the exchange rate $x_i \in [1, M]$, where x_i is the amount of polish zlotys you can buy for 1 Euro. At day i, you may only either trade the whole 1 Euro or do nothing.

Create an algorithm which is $O(\log M)$-competitive.

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