Verification of Infinite-state Systems

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Exploration of infinite-state spaces.

Integration in the system development process

PathStar

Checking a telephone switch.

- One system
- Verification interleaved with design (300 versions)
- Highly concurrent code
- Complex specification (80/200 properties)

Slam

Checking Windows XP drivers.

- Many systems
- Post-mortem verification

- Sequential code
- Simple specification (i.e.,correct locking/unlocking)

Sources of infinity in software systems

Data manipulation: integers, lists, trees, more general pointer structures, . . .

Control structures: procedures , process creation, . . .

Asynchronous communication: unbounded FIFO queues.

Parameters: number of processes, duration of delays . . .

Real-time: discrete or dense domains.

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Reintroduce the abstracted data incrementally by means of predicate abstraction and counterexample-guided abstraction refinement.
(See (perhaps) this course.)

Extended automata: Syntax

Extended automaton = automaton whose transitions are guarded by and operate on data structures.

An extended automaton is a tuple $E = (X, Q, T, G, A)$ where

- $X = \{x_1, \ldots, x_n\}$ is a finite set of variables over sets $V_1, \ldots, V_n$ of values,
- Q is a finite set of control states,
- $T \subseteq Q \times Q$ is a set of transitions or rules,
- G associates to each transition a guard (a predicate over X, the condition under which the transition can be taken),
- A associates to each transition an action (a possibly nondeterministic assignment to X)

Notation for transitions: $q \xrightarrow[a]{g} q'$, where g guard and a action.

Remark: variables over finite sets of values can be encoded into the states.

Extended automata: Semantics

A configuration is a tuple $\langle q, v_1, \ldots, v_n \rangle$, where

- q is a state, and
- $v_1, \ldots, v_n$ is a valuation of $x_1, \ldots, x_n$ (i.e., $v_i \in V_i$ for every $1 \leq i \leq n$).

The transition system $\mathcal{T}_E$ of an extended automaton E has:

- the set of all configurations as nodes, and
- an edge $\langle q, v_1, \ldots, v_n \rangle \longrightarrow \langle q', v'_1, \ldots, v'_n \rangle$ iff E has a transition $q \xrightarrow{\frac{g}{a}} q'$ such that
  - $v_1, \ldots, v_n$ satisfies the guard g, and
  - $v'_1, \ldots, v'_n$ is one of the possible results of applying a to $v_1, \ldots, v_n$.

Some classes of extended automata

Automata	Variables	Transition
Timed automata	clocks (reals)	$q \xrightarrow[c_2:=0]{c_1 \geq 2} q'$
Pushdown automata	stack	$q \xrightarrow[a/ba]{top=a} q'$
(Ext. of) Petri nets	counters (integers)	$q \xrightarrow[x_2:=x_2+x_3]{x_1=0} q'$
FIFO automata	queues	$q \xrightarrow[l_2?a]{l_1 \neq \epsilon} q'$

Networks of extended automata

A network of extended automata (or just a network) is a tuple $\langle E_1, \ldots, E_m \rangle$ of extended automata over the same set of variables X.

The asynchronous product of a network $\langle E_1, \ldots, E_m \rangle$ is the extended automaton having

- the set $Q = Q_1 \times \ldots \times Q_m$ as states, where $Q_1, \ldots, Q_m$ are the sets of states of $E_1, \ldots, E_m$, and
- for every $i \in \{1, \ldots, m\}$, every state $\langle q_1, q_2, \ldots, q_m \rangle \in Q$ and every transition $q_i \xrightarrow[a]{g} q_i'$ of E_i, a transition

$$\langle q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_m \rangle \xrightarrow[a]{g} \langle q_1, \ldots, q_{i-1}, q_i', q_{i+1}, \ldots, q_m \rangle$$

The reachability problem

Let c, c' be two configurations of an extended automaton E. We say that c' is reachable from c if there is a path in $\mathcal{T}_E$ leading from c to c'.

We consider the following problem:

- Given: An extended automaton E, a set I of initial configurations,
  a set D of dangerous configurations.
- Decide: Is some dangerous configuration reachable
  from some initial configuration ?

The sets I and D may be infinite.

Symbolic search

A general framework for the symbolic reachability problem

Let $post(C)$ denote the immediate successors of a (possibly infinite!) set C of configurations

Forward symbolic search

Initialize $C := I$

Iterate $C := C \cup post(C)$ until

$C \cap D \neq \emptyset$; return "reachable", or

a fixpoint is reached; return "non-reachable"

Backward search: exchange I and D, replace *post* by *pre*.

Question: when is symbolic search effective?

(Forward) Symbolic search effective if . . .

. . . there exists a family $\mathcal{C}$ of sets of configurations satisfying the folllowing six properties:

1. each $C \in \mathcal{C}$ has a symbolic finite representation,

2. $I \in \mathcal{C}$,

3. if $C \in \mathcal{C}$, then $C \cup post(C) \in \mathcal{C}$ (and effectively computable),

4. emptiness of $C \cap D$ is decidable,

5. $C_1 = C_2$ is decidable (to check if fixpoint has been reached),, and

6. any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ reaches a fixpoint after finitely many steps.

Remarks

Similar conditions for backward search.

The shape of I is determined by the model.

The shape of D is determined by the specification.

This asymmetry can make one of the two searches far more useful than the other.

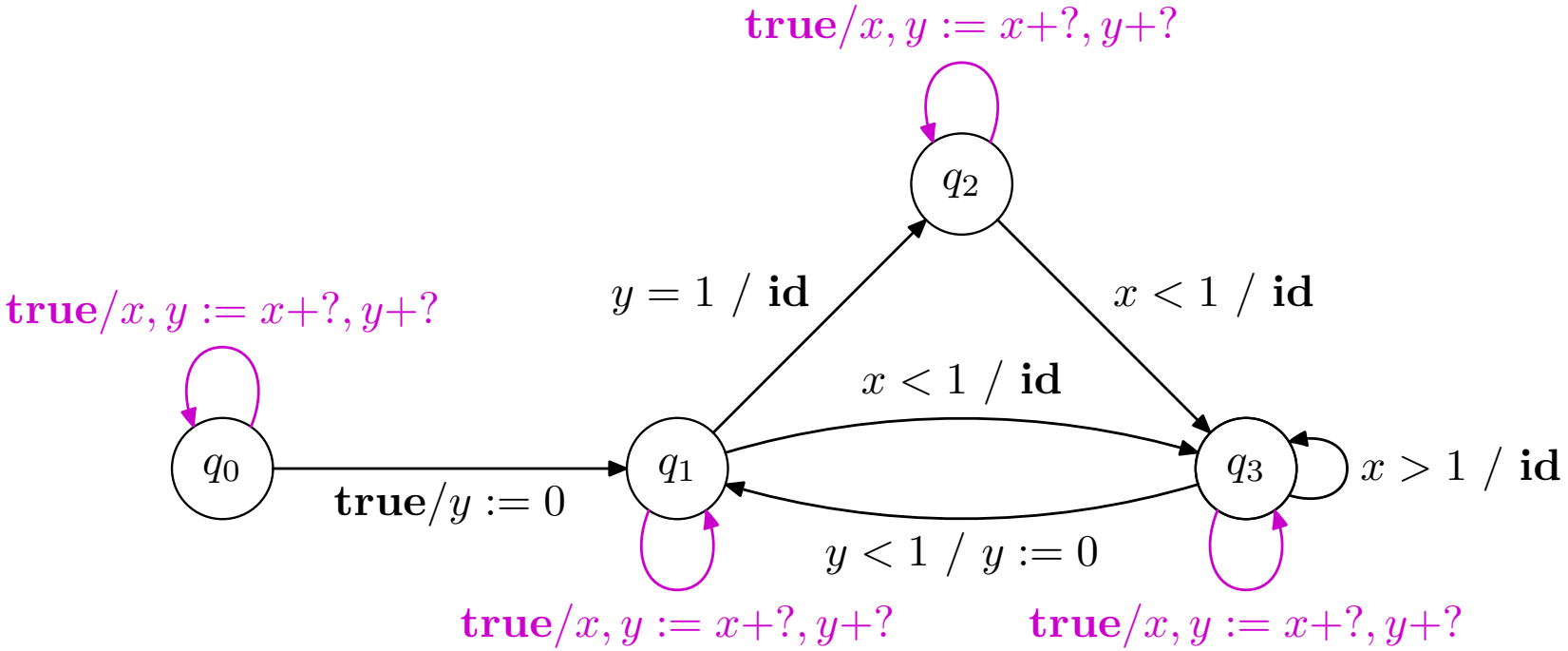
Program for the rest of the course

We consider four classes of systems, and use them to illustrate four different techniques to obtain an effective symbolic search.

- Timed automata: Finite partitions.
- Broadcast protocols: Well quasi-orders.
- Pushdown automata: Accelerations.
- (Lossy) channel systems: Acclerations and Learning.

Timed automata

Timed automata

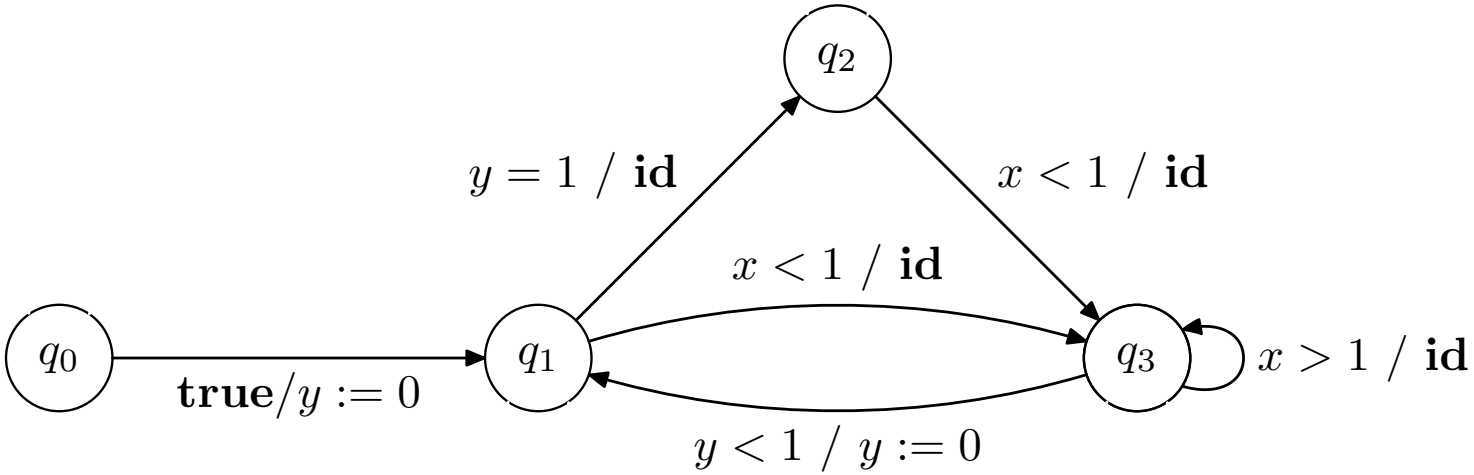


Automata extended with clocks (non-negative real variables).

Time-elapse transitions: self-loops, no guard, the action adds an arbitrary positive real to all clocks (same for all).

Location-switch transitions: guarded by boolean combination of comparisons with integer bounds, the action resets a subset of clocks.

Timed automata



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Case study: Fischer's mutex protocol

A simplified version (so that the analysis can be visualized in one slide).

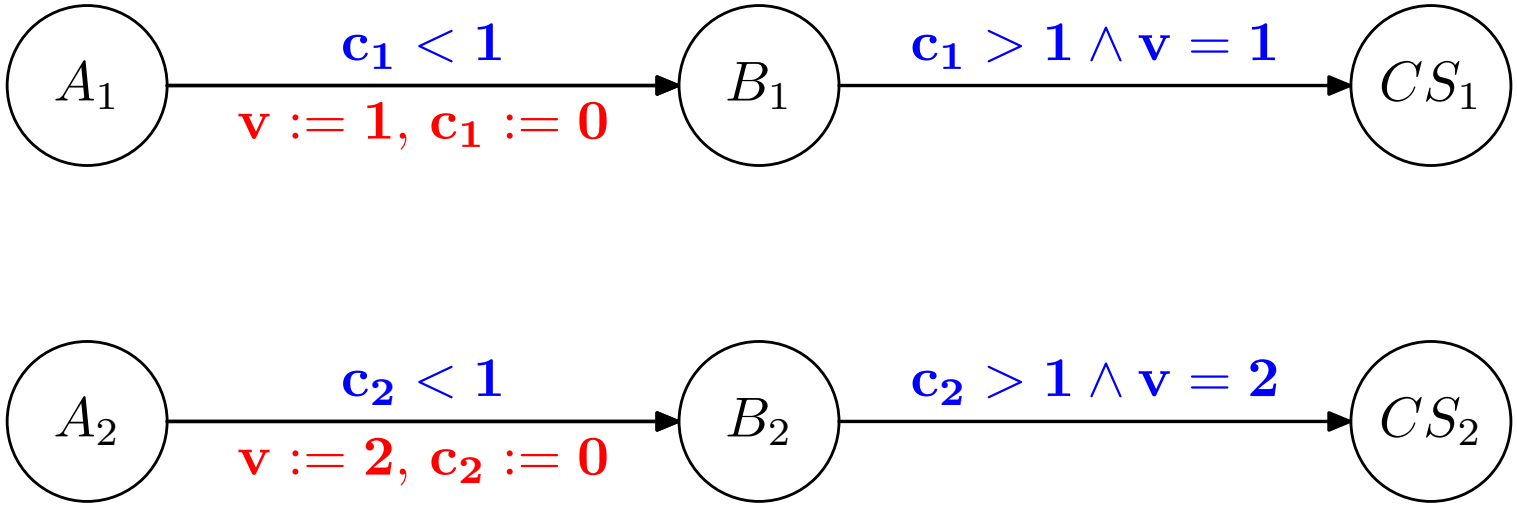
var `v:{1,2}` **init** `1;`

delay `< 1;`	**delay** `< 1;`
`v:= 1;`	`v:= 2;`
delay `> 1;`	**delay** `> 1;`
if `v = 1` **then goto** `cs1`	**if** `v = 2` **then goto** `cs2`

Model

var $v : \{1, 2\}$ **init** 1

var $c_1, c_2 :$ clock **init** 0



Network of 2 timed automata.

Equivalent to one single automaton with 9 states.

Symbolic search for timed automata

The set I of initial configurations is usually of the form

$$\{\langle q, 0, \ldots, 0\rangle \mid q \in Q_I\}$$

The set D of dangerous final configurations is usually of the form

$$\{\langle q, t_1, \ldots, t_n\rangle \mid q \in Q_D \text{ and } t_1, \ldots, t_n \geq 0\}$$

Question: Is reachability decidable for I and D of this form?

Regions

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Let *max* be the maximal constant appearing in the syntactic description of the automaton

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$$x_i \leq k \qquad \text{or} \qquad x_i \geq k \qquad \text{or} \qquad x_i - x_j \leq k$$

where $k \in \{0, 1, \ldots, max\}$.

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Two configurations $\langle q, \mathbf{t} \rangle$ and $\langle r, \mathbf{u} \rangle$ are equivalent, denoted by $\langle q, \mathbf{t} \rangle \sim \langle r, \mathbf{u} \rangle$, if

- $q = r$, and
- for every constraint $\gamma \in \Gamma$: $\mathbf{t}$ satisfies γ iff $\mathbf{u}$ satisfies γ.

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An equivalence class of configurations is called a region.

Characterizing regions

Given a real number z, let $\lfloor z \rfloor$ denote its integer and $\underline{z}$ its fractional part.

$\langle q, \mathbf{t} \rangle \sim \langle r, \mathbf{u} \rangle$ holds iff $q = r$ and for every $i, j \in \{0, 1, \ldots, max\}$:

(a) $\lfloor t_i \rfloor = \lfloor u_i \rfloor$ or $t_i > max$ and $u_i > max$,

(because $k - 1 \leq t_i \leq k$ iff $k - 1 \leq u_i \leq k$ for all $k \in \{1, \ldots, max\}$)

(b) if $t_i, u_i \leq max$, then $\underline{t_i} = 0$ iff $\underline{u_i} = 0$,

(because $k \leq t_i \leq k$ iff $k \leq u_i \leq k$ for all $k \in \{0, \ldots, max\}$))

(c) if $t_i, u_i, t_j, u_j \leq max$, then $\underline{t_i} < \underline{t_j}$ iff $\underline{u_i} < \underline{u_j}$.

(because of (a), (b), and $t_i - t_j \leq 0$ iff $u_i - u_j \leq 0$)

Example: $\langle q \;\; 3.2 \;\; 4.7 \;\; 3.5 \rangle \sim \langle q \;\; 3.7 \;\; 4.9 \;\; 3.8 \rangle$

$\langle q \;\; 3.2 \;\; 4.7 \;\; 3.5 \rangle \not\sim \langle q \;\; 3.2 \;\; 4.7 \;\; 3.9 \rangle$

Two observations

The number of regions is bounded by $(2max + 2)^n \cdot n! \cdot 2^n$ (exercise).

- Exponential in both the number of clocks n and in the maximal constant max when written in binary.

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Two equivalent configurations enable exactly the same transitions.

- Because they satisfy exactly the same guards.

Effectiveness of forward and backward search

We choose $\mathcal{C}$ as the powerset of the set of regions.

Theorem [Alur, Dill, TCS 1994]:
Both forward and backward search satisfy conditions (1) - (6).

Proof for forward search in the next slides, for backward search analogous.

Proof

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   and so $\{\langle q, 0, \ldots, 0\rangle\}$ is a region for each state q.

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   Since $\langle r, \mathbf{u}\rangle \in post(C)$, there is $\langle q, \mathbf{t}\rangle \in C$ such that $\langle q, \mathbf{t}\rangle \longrightarrow \langle r, \mathbf{u}\rangle$.
   We consider the cases of time-elapse and location-switch transitions
   separately.

Time-elapse transitions (“proof by example”):

$$\langle\ q \quad t_1 \quad t_2 \quad t_3\ \rangle \xrightarrow{\lfloor\tau\rfloor + \underline{\tau}} \langle\ r \quad u_1 \quad u_2 \quad u_3\ \rangle$$

$$\sim$$

$$\langle\ r \quad u'_1 \quad u'_2 \quad u'_3\ \rangle$$

Time-elapse transitions (“proof by example”):

$$\begin{array}{c} 0 < \underline{t_1} < \underline{t_2} < \underline{t_3} < 1 \\ \langle \quad q \quad t_1 \quad t_2 \quad t_3 \quad \rangle \end{array} \xrightarrow{\lfloor \tau \rfloor + \underline{\tau}} \begin{array}{c} 0 = \underline{u_2} < \underline{u_3} < \underline{u_1} < 1 \\ \langle \quad r \quad u_1 \quad u_2 \quad u_3 \quad \rangle \end{array}$$

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\langle\ q \quad 3.1 \quad 1.5 \quad 2.7\ \rangle & \xrightarrow{1 + 0.5} & \langle\ r \quad 4.7 \quad 3.0 \quad 4.3\ \rangle \\
 & & \sim \\
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\sim \\
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\langle\ q \quad t'_1 \quad t'_2 \quad t'_3\ \rangle & \xrightarrow{\lfloor\tau\rfloor + \delta} & \langle\ r \quad u'_1 \quad u'_2 \quad u'_3\ \rangle \\
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\sim & & \sim \\
\langle\ q \quad 3.05 \quad 1.25 \quad 2.95\ \rangle & \xrightarrow{1 + 0.75} & \langle\ r \quad 4.8 \quad 3.0 \quad 4.7\ \rangle \\
\langle\ q \quad t'_1 \quad t'_2 \quad t'_3\ \rangle & \xrightarrow{\lfloor\tau\rfloor + \delta} & \langle\ r \quad u'_1 \quad u'_2 \quad u'_3\ \rangle \\
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 & \sim & & & & & \sim & & \\
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\langle\ q & t'_1 & t'_2 & t'_3\ \rangle & \xrightarrow{\lfloor\tau\rfloor + \delta} & \langle\ r & u'_1 & u'_2 & u'_3\ \rangle \\
 & 0 < \underline{t'_1} < \underline{t'_2} < \underline{t'_3} < 1 & & & \underline{u'_3} < \delta < \underline{u'_1} & & 0 = \underline{u'_2} < \underline{u'_3} < \underline{u'_1} < 1 &
\end{array}
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Location-switch transitions (“proof by example”):

$$\langle\ r \quad t_1 \quad t_2 \quad t_3\ \rangle \xrightarrow{x_2 := 0} \langle\ r \quad t_1 \quad 0 \quad t_3\ \rangle$$

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$$\langle \; r \quad 3.3 \quad 0 \quad 2.4 \; \rangle$$

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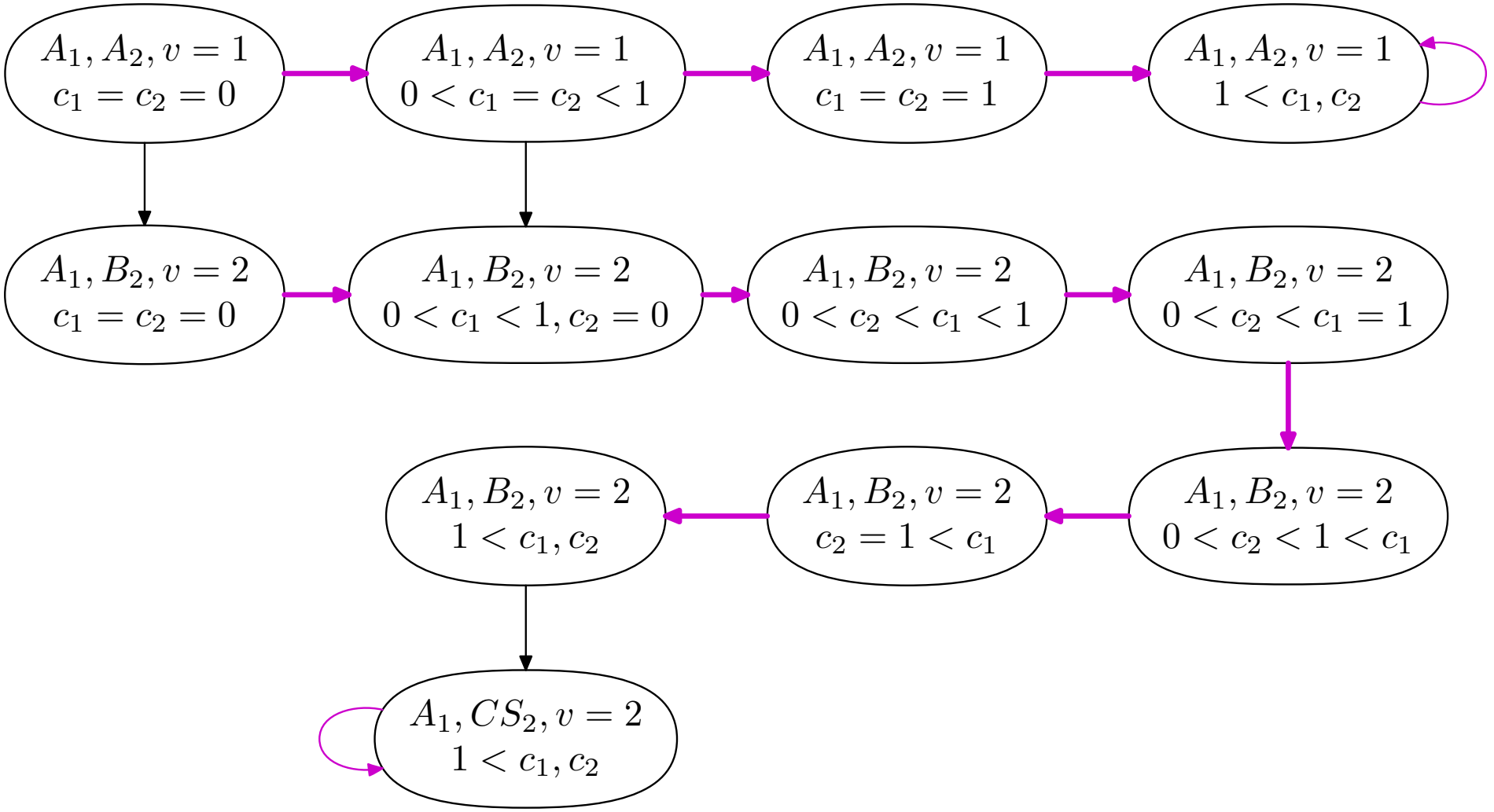
(One half of) The region graph of Fischer's protocol



Complexity of the reachability problem

The reachability problem is PSPACE-complete.

Reason: exponential dependence in the number of clocks or the size of max is unavoidable.

The problem remains PSPACE-hard if the constants or the number of clocks (but not both) are bounded.

Repeated reachability for timed automata

A control state is repeatedly reachable if some non-zeno infinite execution containing infinitely many location-switch transitions visits the control state infinitely often.

The repeated reachability problem can be solved easily using the region graph.

To know more

Tutorial slides by Rajeev Alur, available at
http://www.cis.upenn.edu/ alur/talks.html

Check the publications of: Alur, Asarin, Bouyer, Courcoubetis, Dill, Henzinger, Laroussinie, Larsen, Maler, Sifakis, Wilke . . . .

UPPAAL is a popular tool for verification of timed automata,
http://www.uppaal.com/

Broadcast protocols

Broadcast protocols

Introduced by Emerson and Namjoshi in LICS '98.

All processes execute the same algorithm, i.e., all finite automata are identical.

Processes are indistinguishable (no IDs).

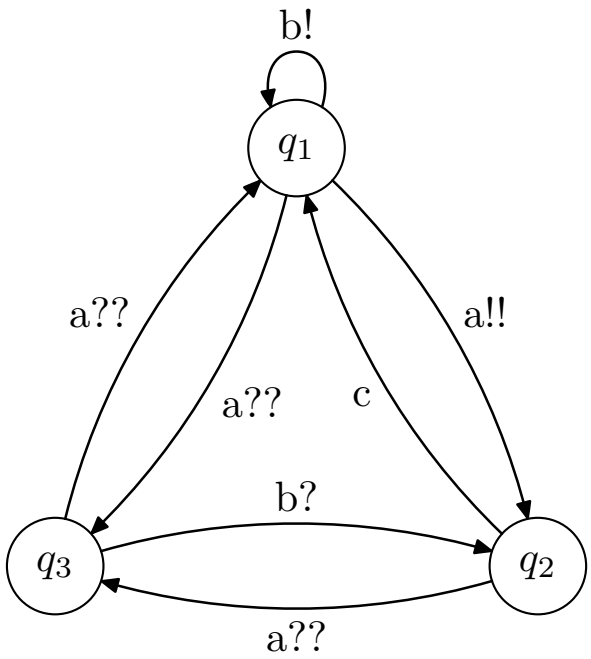
Communication mechanisms:

Rendezvous: two processes exchange a message and move to new states.

Broadcasts: a process sends a message to all others,
all processes move to new states.

We introduce syntax and semantics and show translation into extended automata.

Syntax



a!! : broadcast a message along (channel) *a*

a?? : receive a broadcasted message along *a*

b! : send a message to one process along *b*

b? : receive a message from one process along *b*

c : change state without communicating with anybody

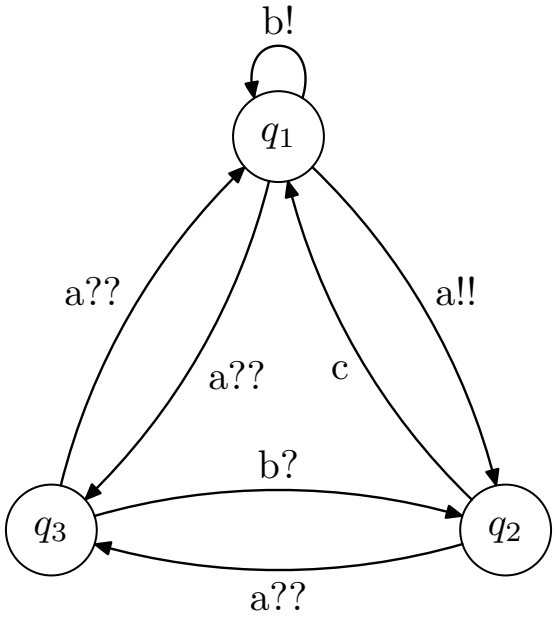
Semantics

The global state of a broadcast protocol is completely determined by the number of processes in each state.

Configuration: mapping $c\colon Q \to \mathbb{N}$

represented by the vector $(c(q_1), \ldots, c(q_n))$.

Semantics for a given initial configuration: finite transition system with configurations as nodes.



$$(3,1,2) \longrightarrow (4,0,2) \quad \text{(silent move } c\text{)}$$

$$(3,1,2) \longrightarrow (3,2,1) \quad \text{(rendezvous } b\text{)}$$

$$(3,1,2) \longrightarrow (2,1,3) \quad \text{(broadcast } a\text{)}$$

$$(185,3425,17) \longrightarrow (17,1,3609) \quad \text{(broadcast } a\text{)}$$

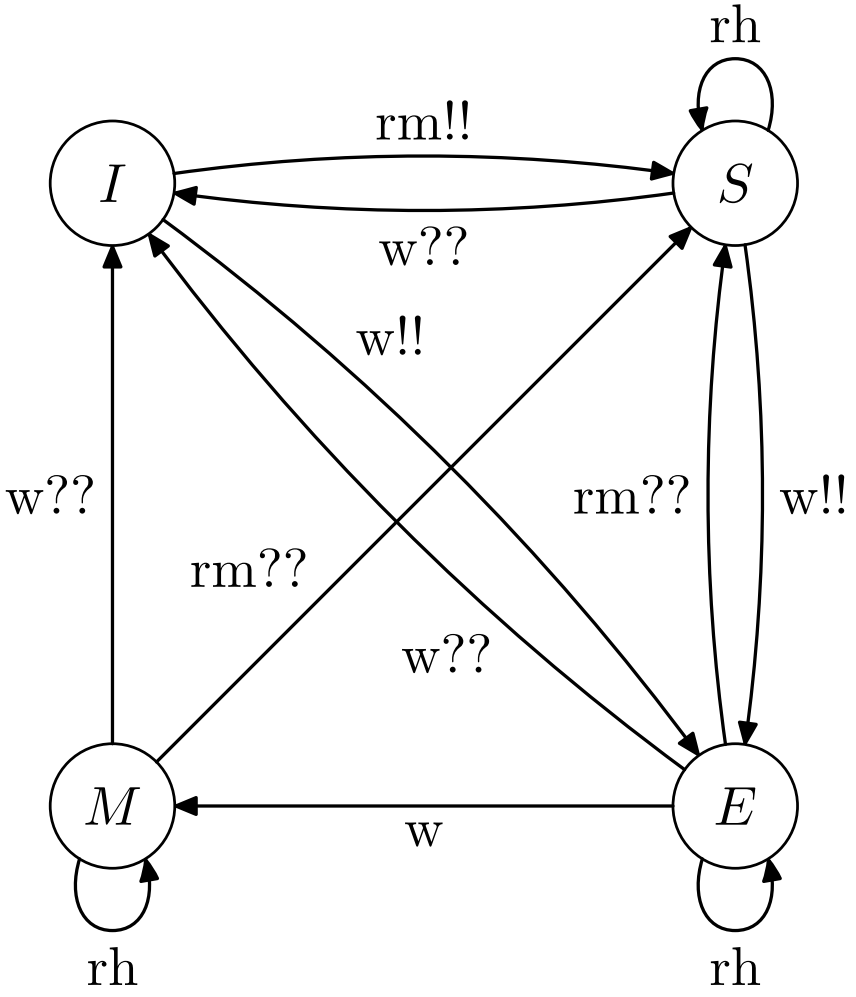
Parametrized configuration: partial mapping $p : Q \to \mathbb{N}$.

- Intuition: “configuration with holes”.
- Formally: set of configurations (total mappings matching p).

Infinite transition system of the broadcast protocol:

- Fix an initial parametrized configuration p_0.
- Take the union of all finite transition systems for each configuration $c \in p_0$.

Case study: A MESI cache-coherence protocol



rh : read hit

rm : read miss

w : write hit/write miss

Broadcast protocols as extended automata

We translate the MESI-protocol into an extended automaton.

We take:

- One (non-negative) integer variable per state of the protocol: m, e, s, i.
- One single control state q.
- One transition $q \xrightarrow[a]{g} q$ for each send transition or silent move of the protocol, see next slide.

A configuration $(n_1, \ldots, n_k)$ of a broadcast protocol corresponds to the configuration $\langle q, n_1, \ldots, n_k \rangle$ of the extended automaton.

Transition	Guard	Action
$I \xrightarrow{rm!!} S$	$i \geq 1$	$m' = m \quad e' = 0 \quad s' = m + e + s + 1 \quad i' = i - 1$
$I \xrightarrow{w!!} E$	$i \geq 1$	$m' = 0 \quad e' = 1 \quad s' = 0 \quad i' = m + e + s + i - 1$
$S \xrightarrow{w!!} E$	$s \geq 1$	$m' = 0 \quad e' = 1 \quad s' = 0 \quad i' = m + e + s + i - 1$
$S \xrightarrow{rh} S$	$s \geq 1$	$m' = m \quad e' = e \quad s' = s \quad i' = i$
$E \xrightarrow{w} M$	$e \geq 1$	$m' = m + 1 \quad e' = e - 1 \quad s' = s \quad i' = i$
$E \xrightarrow{rh} E$	$e \geq 1$	$m' = m \quad e' = e \quad s' = s \quad i' = i$
$M \xrightarrow{rh} M$	$m \geq 1$	$m' = m \quad e' = e \quad s' = s \quad i' = i$

Reachability in broadcast protocols

Typical set I of initial configurations: parametrized configuration.

Typical set D of final configurations: upward-closed sets.

- U is an upward-closed set of configurations if

$$c \in U \text{ and } c' \geq c \text{ implies } c' \in U$$

  where $\geq$ is the pointwise order on $\mathbb{N}^n$.

- Example: states M and S of MESI protocol should be mutually exclusive

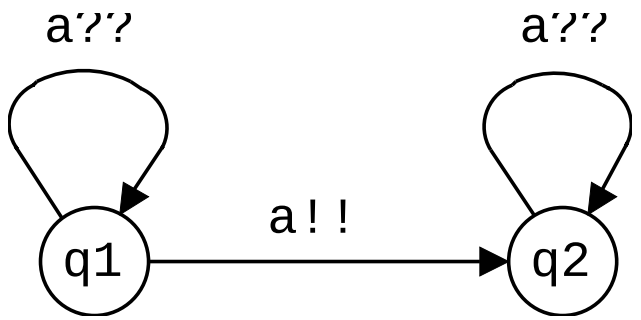
$$D = \{(m, e, s, i) \mid m \geq 1 \wedge s \geq 1\}$$

Question: Is reachability decidable if I is a parametric configuration and D is an upward-closed set?

First try: Forward search

Since $I \in \mathcal{C}$ is required by condition (2), the family $\mathcal{C}$ must contain all parametrized configurations.

Satisfies (1) - (5) but not (6). Termination fails in very simple cases.



$$(\sqcup, 0) \xrightarrow{a} (\sqcup, 1) \xrightarrow{a} (\sqcup, 2) \xrightarrow{a} \ldots$$

Second try: Backward search

Since $D \in \mathcal{C}$ is required by condition (2), the family $\mathcal{C}$ must contain all upward-closed sets.

Theorem [Abdulla *et al.*, I&C 160, 2000], [E. *et al.*, LICS'99]
Backward search satisfies conditions (1) - (6).

Proof in the next slides.

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   Pick some minimal element $m_1 \in U_1$.
   Pick for every $i > 1$ some minimal element $m_i \notin U_1 \cup \ldots \cup U_{i-1} = U_{i-1}$.
   Consider the sequence $m_1, m_2, m_3, \ldots$

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Complexity

Consider the sequences $C = c_1, c_2, c_3, \ldots$, where $c_i \in \mathbf{N}^k$ for all $i \geq 1$, that satisfy:

- $c_1 \leq (1, \ldots, 1)$, and
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By Dickson's lemma any such sequence contains indices i, j such that $c_i \leq c_j$.

Let $J(C)$ be the smallest j for which such an i exist.

Let $G(k)$ be the maximum over all C's of the index $J(C)$.

How fast can G grow?

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Backward search may need a non-primitive recursive number of iterations.

However: Still useful in practice!

Application to the MESI-protocol

Are the states M and S mutually exclusive?

Check if the upward-closed set with minimal element

$$m = 1,\ e = 0,\ s = 1,\ i = 0$$

can be reached from the initial parametrized configuration

$$m = 0,\ e = 0,\ s = 0,\ i = \sqcup$$

Proceed as follows:

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Case studies

Other cache-coherence protocols: Berkeley RISC, Illinois, Xerox PARC Dragon, DEC Firefly, Futurebus +, etc.

[Delzanno, FMSD'03]:

- Model extended with more complicated guards.
- Termination guarantee gets lost.
- Upward-closed sets represented by linear constraints.
- Backward-search algorithm must be refined: Possibly more iterations, but each iteration has lower complexity.

[Emerson,Kahlon, CHARME'03,TACAS'03]:

- Restricted models still able to model the cache-coherence protocols.
- Much faster algorithms.

Symbolic search for other models

Lossy channel systems [Abdulla and Jonsson, I&C '93], [Abdulla et al, CAV'98].

- Configuration: $\langle q, \mathbf{w} \rangle$, where q state and $\mathbf{w} = (w_1, \ldots, w_n)$ vector of words representing the current queue contents
- Family $\mathcal{C}$: upward-closed sets with respect to the subsequence order

  $abba \leq bbaabaaabbabb$

  Dickson's lemma $\rightarrow$ Higman's lemma
- Backward search satisfies (1) - (6).

Timed Petri nets [Abdulla and Nylén, ICATPN'01].

- Configuration: $\langle q, B \rangle$, where B finite bag of vectors of reals.
- Family $\mathcal{C}$: existential zones.

Repeated reachability in broadcast protocols

The following problem is undecidable:

Given: a broadcast protocol,
an initial parametrized configuration $p = (\sqcup, 0, \ldots, 0)$

To decide: is there an integer n such that the transition system
with $(n, 0, \ldots, 0)$ as initial configuration
has an infinite computation ?

Can be reformulated as a repeated reachability problem where
$I = (\sqcup, 0, \ldots, 0)$ and D = set of all configurations.

Pushdown automata

Pushdown automata

Automata extended with one stack.

Transitions:

- Guards: check the topmost symbol in the stack.
- Actions: replace the topmost symbol by a fixed word.
- Notation: $\langle p, \gamma \rangle \hookrightarrow \langle p', v \rangle$
- Normalization: $|v| \leq 2$.

We use P, Γ, Δ for the sets of control states, stack symbols, and rules, respectively.

Configurations: pairs $\langle p, w \rangle$, where p is a control state and w is a word.
(Stack, topmost symbol is the first letter.)

PDAs as models of sequential programs

Programs determined by:

- Control flow: assignments, conditionals, loops ,
  procedure calls with parameters/return values.
- Local variables of each procedure.
- Global variables.

State space determined by:

- Program pointer.
- Values of global variables.
- Values of local variables (of current procedure).
- Activation records (return addresses, copies of locals).

Interpretation of a configuration $\langle q, \gamma v \rangle$:

q holds values of global variables.

γ holds (program pointer, values of local variables).

v holds stack of (return address, saved locals).

Restriction: finite datatypes.

Correspondence between statements and rules:

$\langle q, \gamma \rangle \hookrightarrow \langle q', \gamma' \rangle$	simple statement
$\langle q, \gamma \rangle \hookrightarrow \langle q', \gamma' \gamma'' \rangle$	procedure call
$\langle q, \gamma \rangle \hookrightarrow \langle q', \epsilon \rangle$	return statement

Case study: Drawing skylines

```
void m() {
    if (?) {
        s(); right();
        if (?) m();
    } else {
        up(); m(); down();
    }
}
```

```
void s() {
    if (?) return;
    up(); m(); down();
}

main() {
  s();
}
```

Model

```
void s() {
    s0: if (?) s1: return;
    s2: up();
    s3: m();
    s4: down(); s5:
}
```

var st : **stack** of $\{s_0, \ldots, s_5, \ldots\}$

$\langle p, s_0 \rangle \hookrightarrow \langle p, s_2 \rangle \quad \langle p, s_0 \rangle \hookrightarrow \langle p, \epsilon \rangle$

$\langle p, s_2 \rangle \hookrightarrow \langle p, up_0\ s_3 \rangle$

$\langle p, s_3 \rangle \hookrightarrow \langle p, m_0\ s_4 \rangle$

$\langle p, s_4 \rangle \hookrightarrow \langle p, down_0\ s_5 \rangle \quad \langle p, s_5 \rangle \hookrightarrow \langle p, \epsilon \rangle$

Symbolic reachability in pushdown automata

A set of configurations C is regular if for every control point p, the set $\{w \in \Gamma^* \mid \langle p, w \rangle \in C\}$ is regular.

Typically, I and D are regular sets of configurations.
(Even very simple ones, like $\langle p, \Gamma^* \rangle$.)

Family $\mathcal{C}$: regular sets

Backward search: Do conditions (1) - (6) hold ?

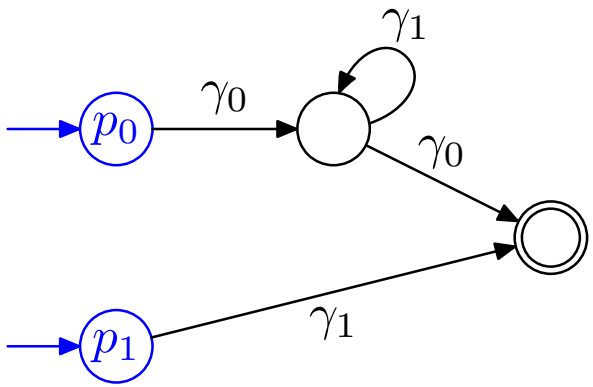
1. Each regular set can be finitely represented by a NFA. $\surd$

NFA for a pushdown system:

- P as set of initial states and Γ as alphabet.
- $\langle p, v \rangle$ recognized if $p \xrightarrow{v} q$ for some final state q.

Example: $P = \{p_0, p_1\}$ and $\Gamma = \{\gamma_0, \gamma_1\}$

Automaton coding the set $\langle p_0, \gamma_0 \gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1 \rangle$:



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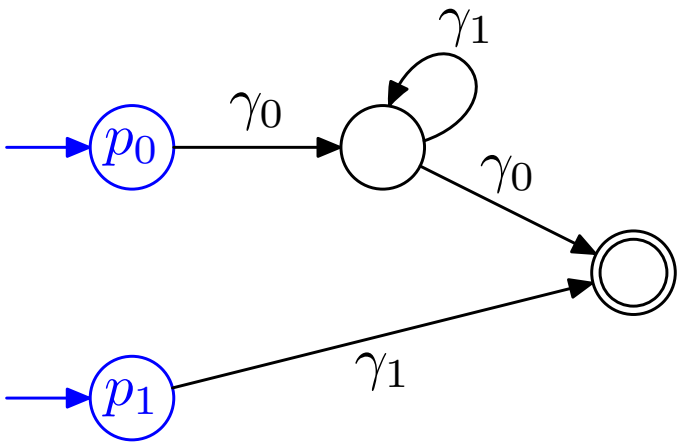

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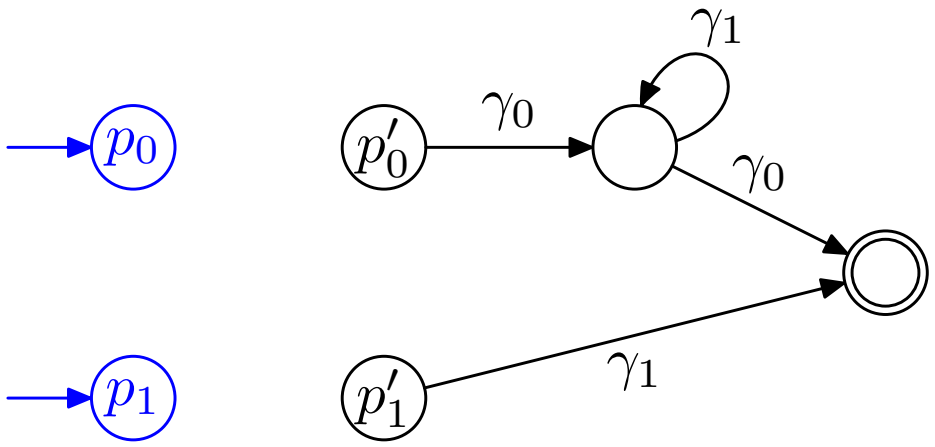

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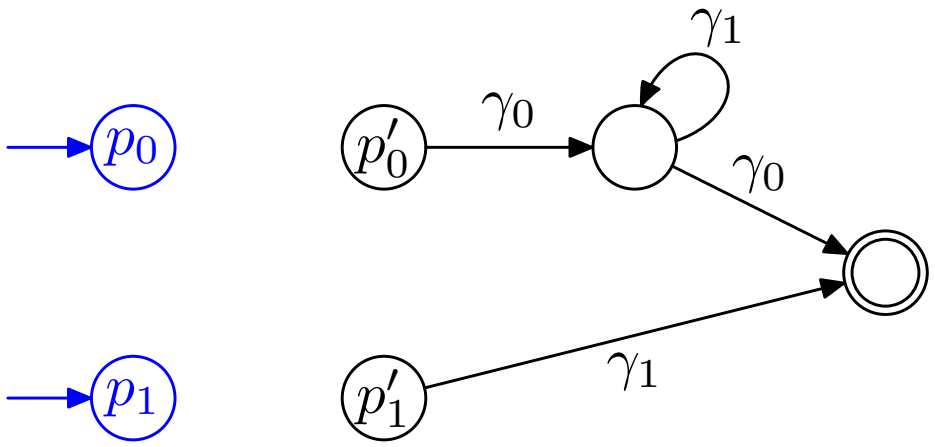

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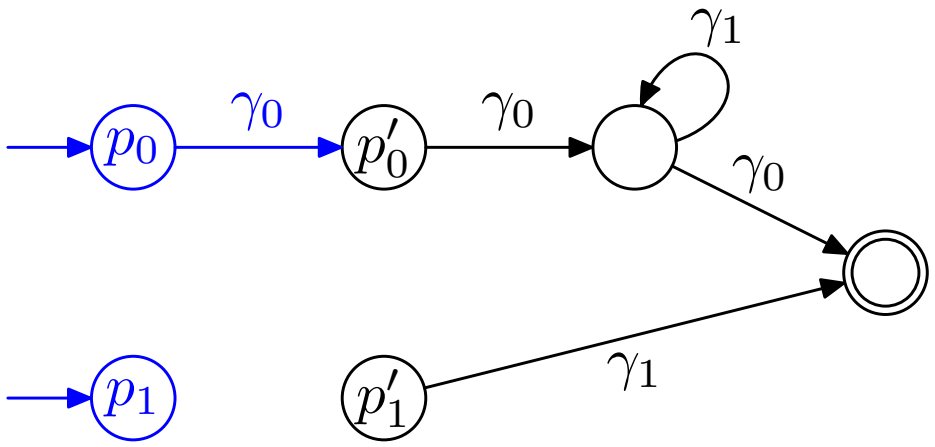

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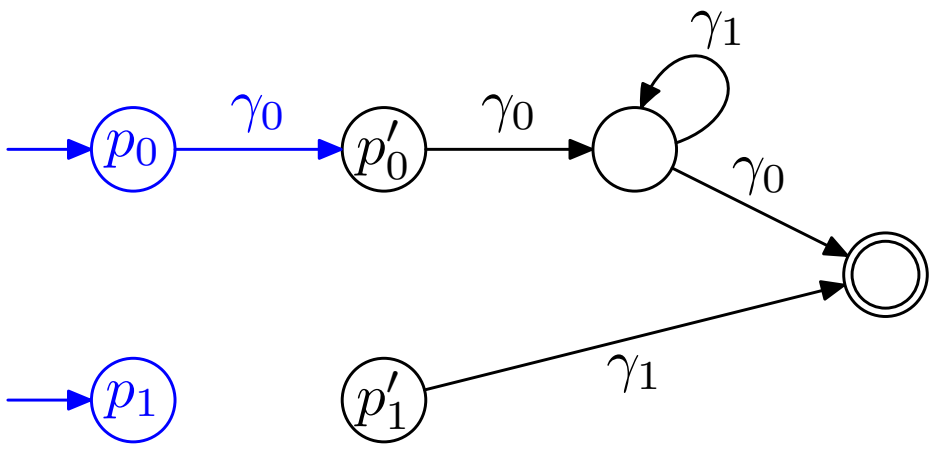

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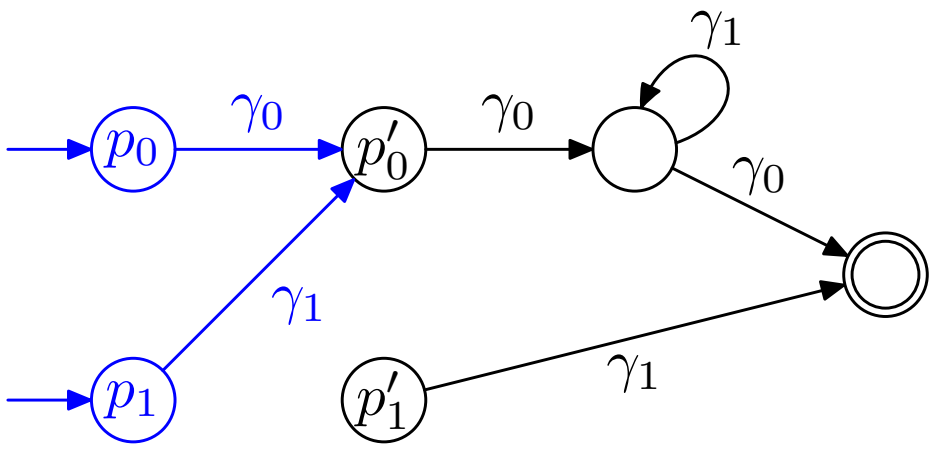

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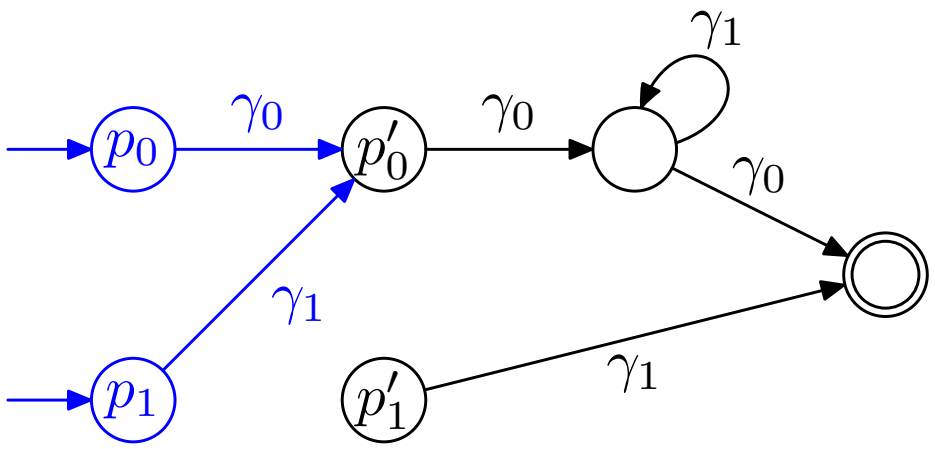

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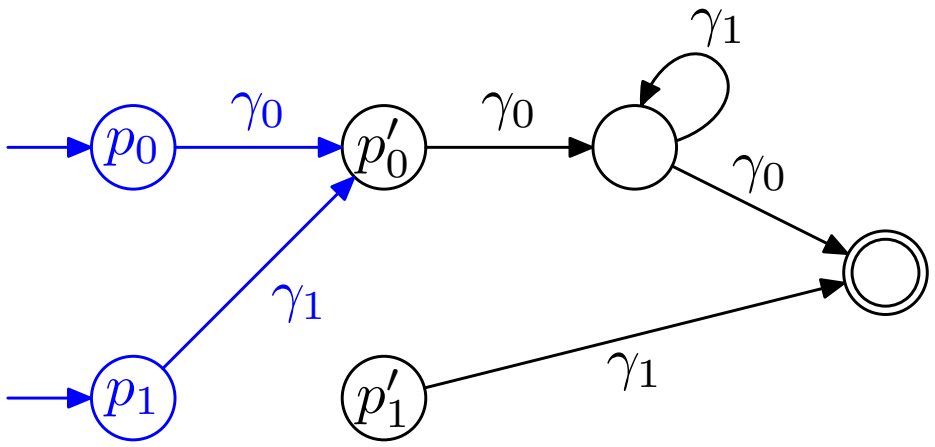

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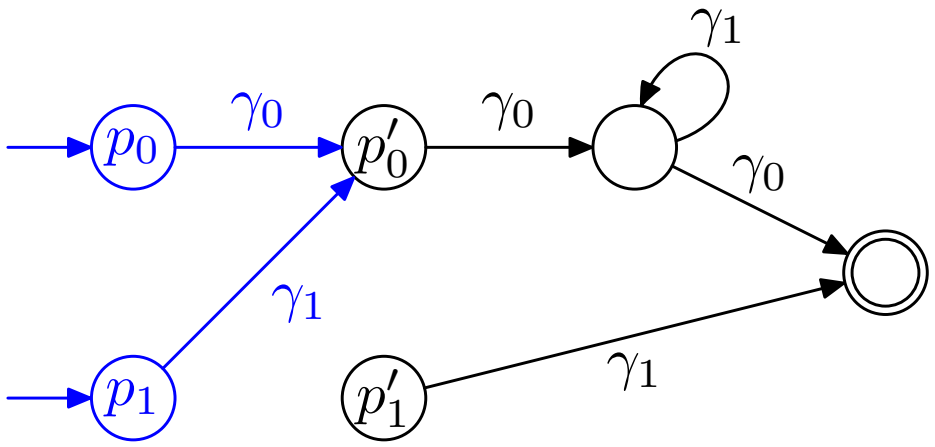

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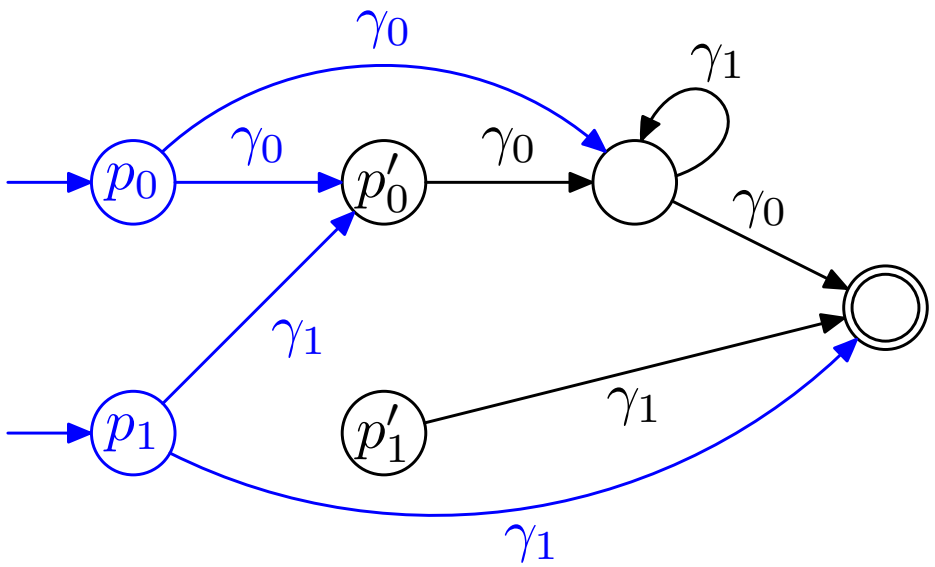

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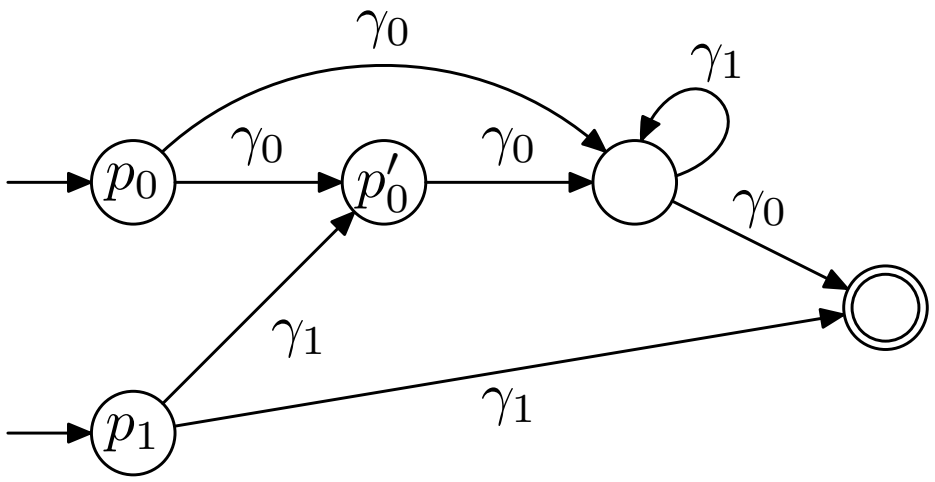

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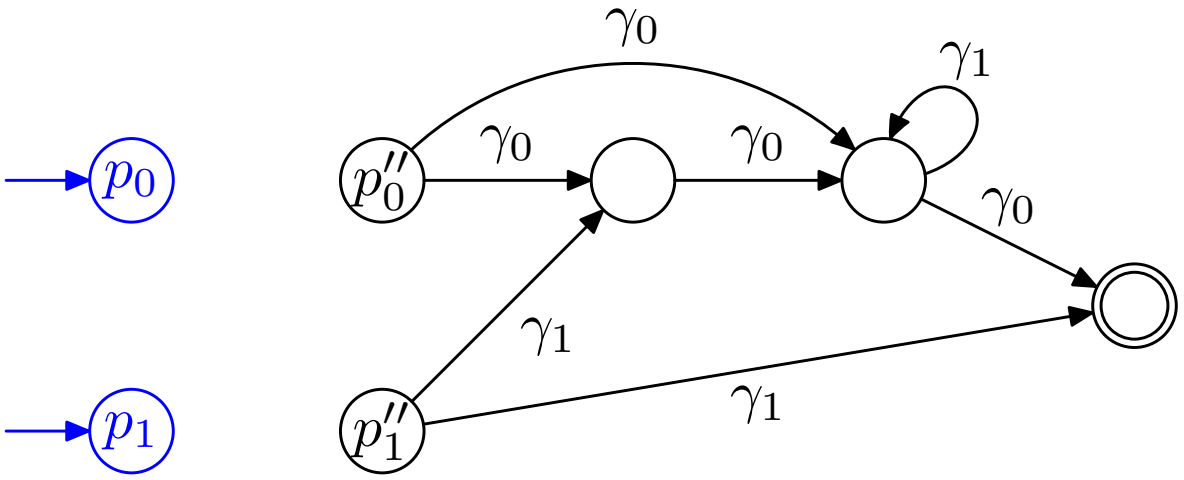

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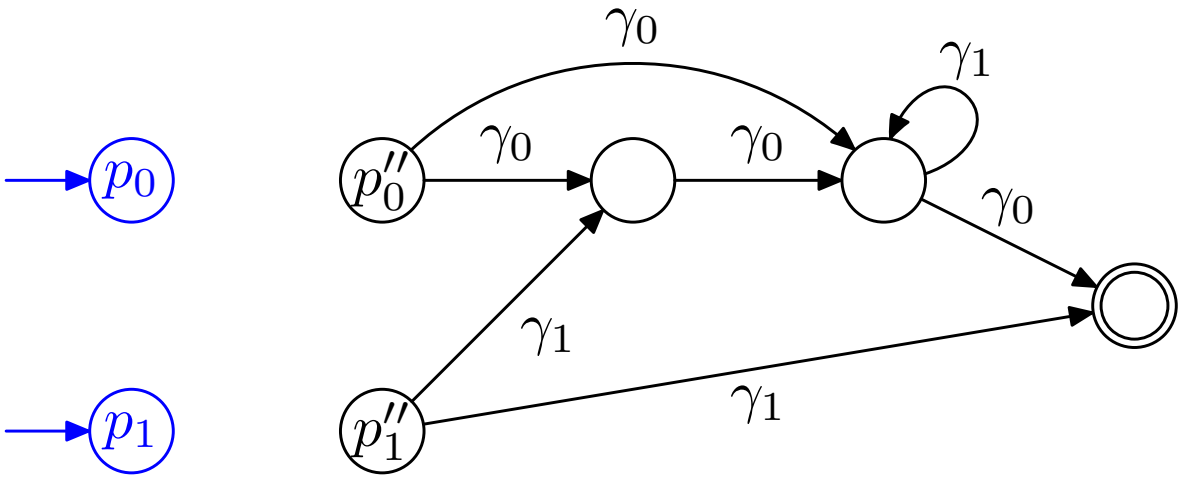

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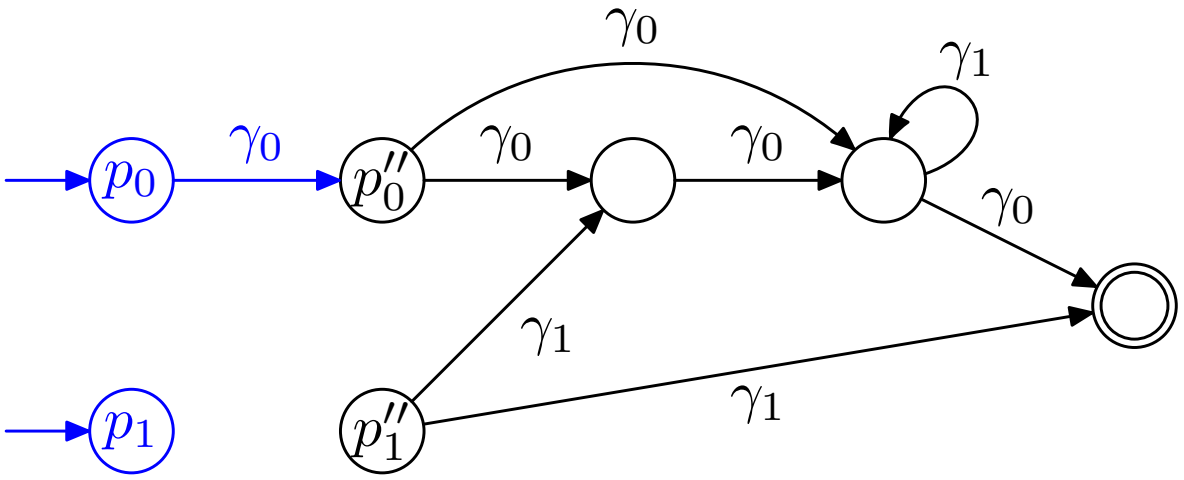

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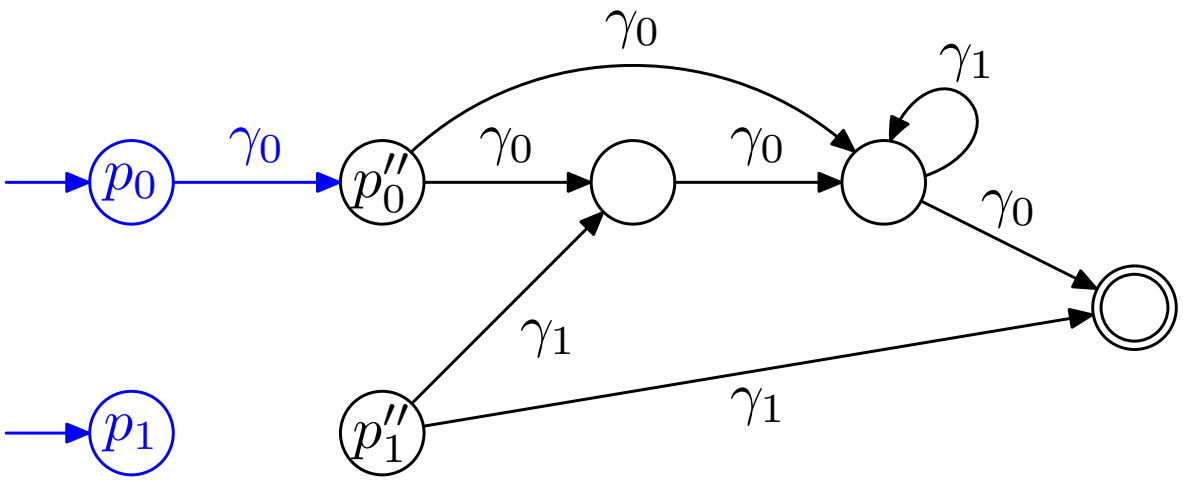

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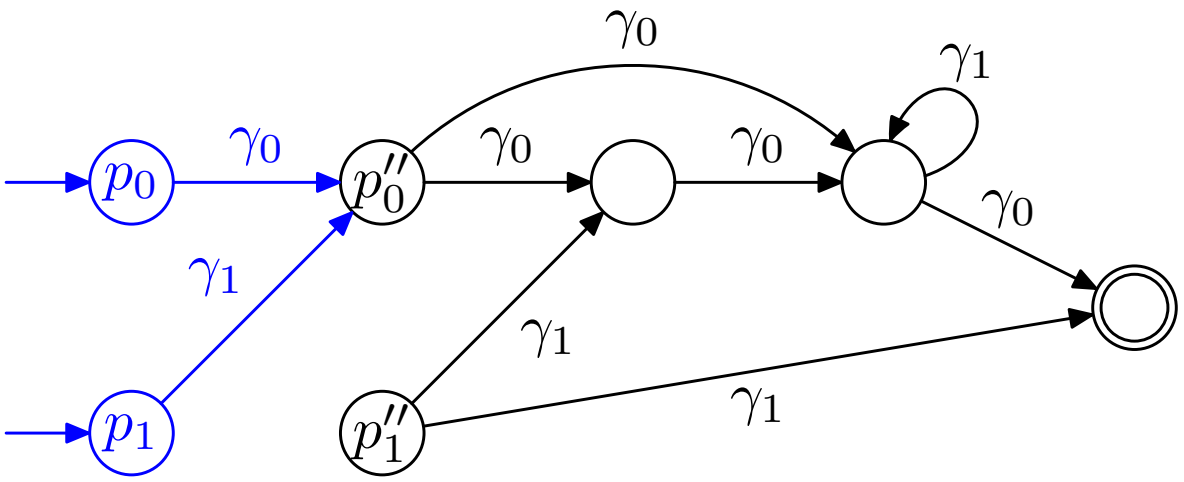

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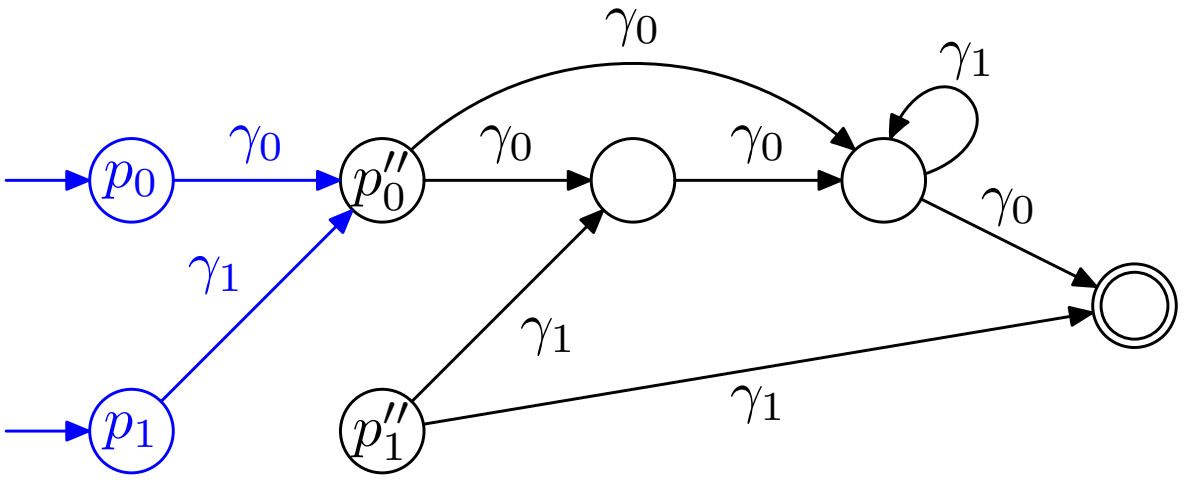

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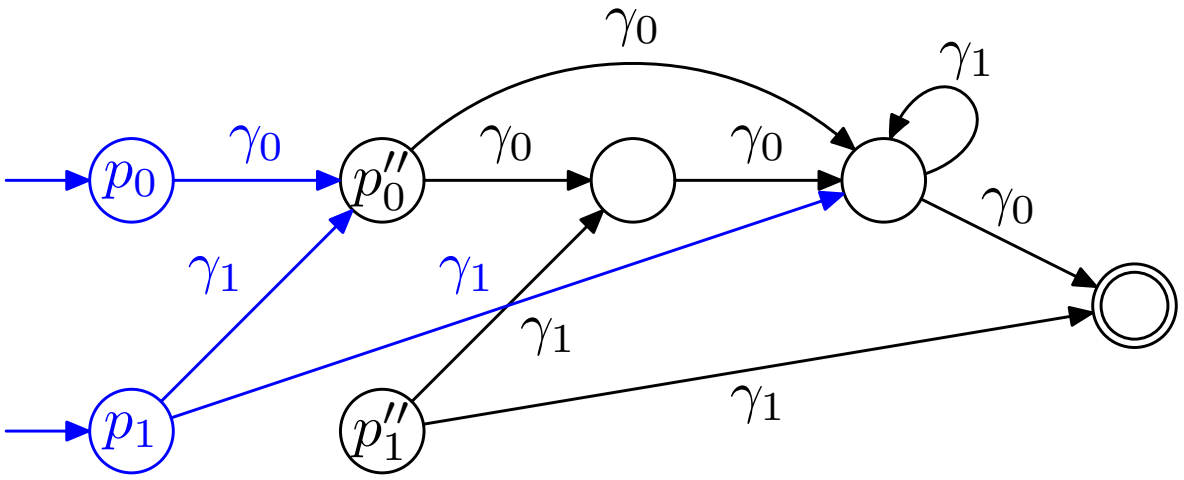

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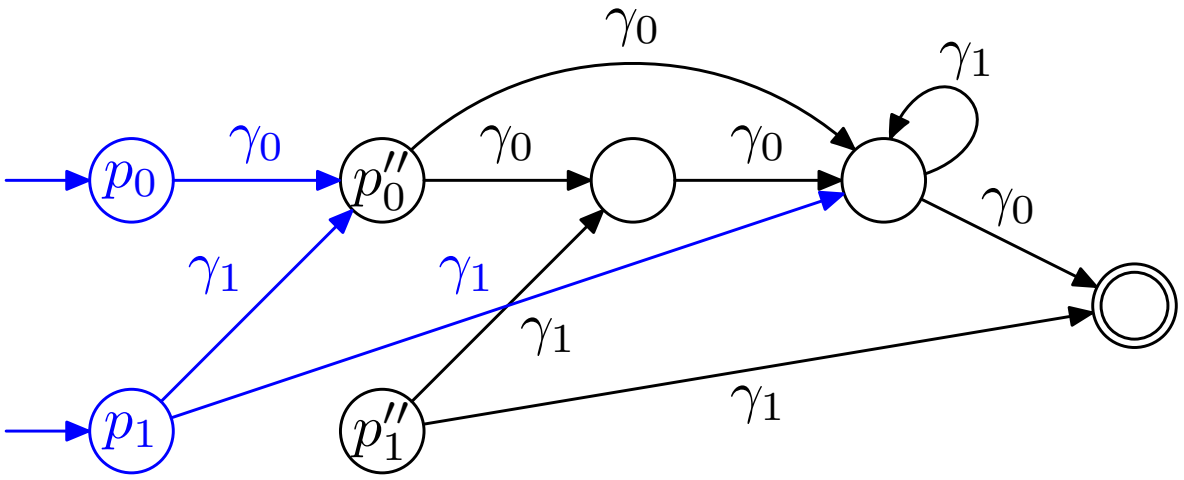

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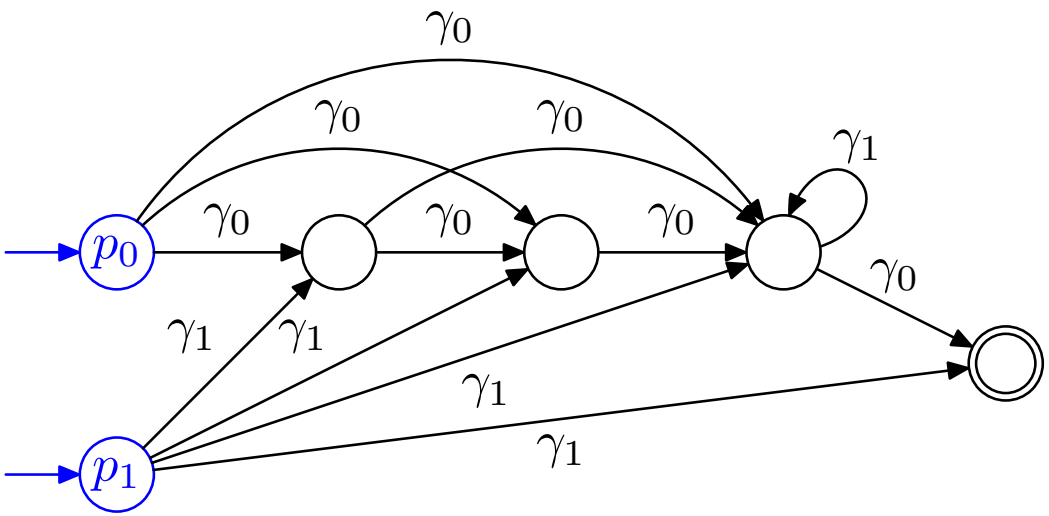

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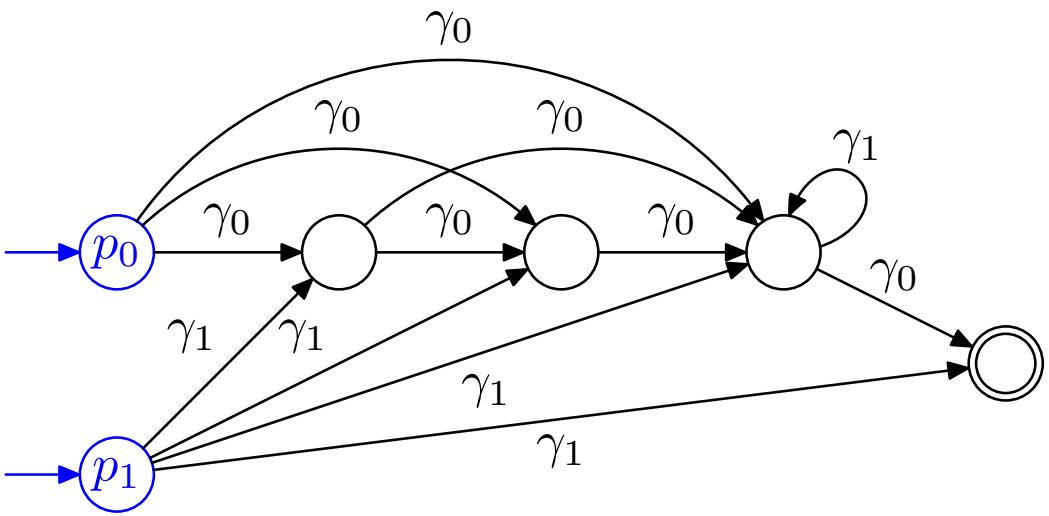
$$\Delta = \{ \langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_0, \epsilon \rangle \ , \ \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_0, \epsilon \rangle \ , \ \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle \ \}$$



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5. $C_1 = C_2$ is decidable. $\surd$

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6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint. FAILS!

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 & \ldots & &
\end{array}
$$

However, the fixpoint

$$\begin{aligned} pre^*(D) &= \langle p_0, \gamma_0^+ \gamma_1^* \gamma_0 \rangle \cup \\ &\quad \langle p_1, \gamma_1 \gamma_0^* \gamma_1^* (\epsilon + \gamma_0) \rangle \end{aligned}$$

is regular.

How can we compute it?

Accelerations

By definition, $pre(D) = \bigcup_{i \geq 0} C_i$
where $C_0 = D$ and $C_{i+1} = C_i \cup pre(C_i)$ for every $i \geq 0$

If convergence fails, try to compute an acceleration :
a sequence $D_0 \subseteq D_1 \subseteq D_2 \ldots$ such that

(a) $\forall i \geq 0 : C_i \subseteq D_i$

(b) $\forall i \geq 0 : D_i \subseteq \bigcup_{j \geq 0} C_j = pre(D)$

Property (a) ensures capture of (at least) the whole set $pre(D)$

Property (b) ensures that only elements of $pre(D)$ are captured

The acceleration guarantees termination if

(c) $\exists i \geq 0 : D_{i+1} = D_i$

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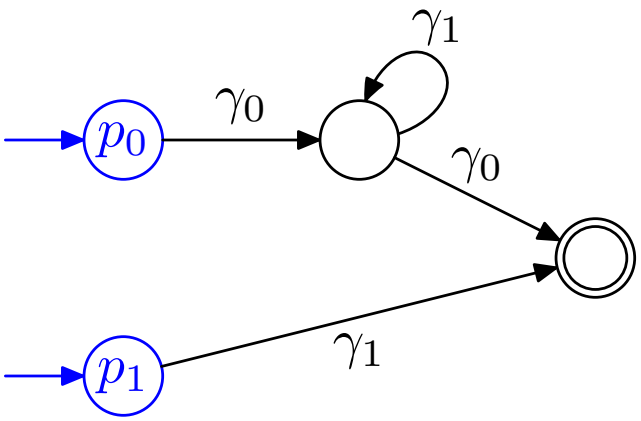

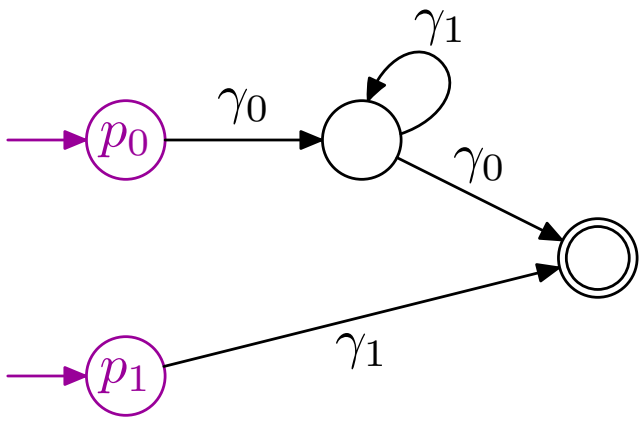

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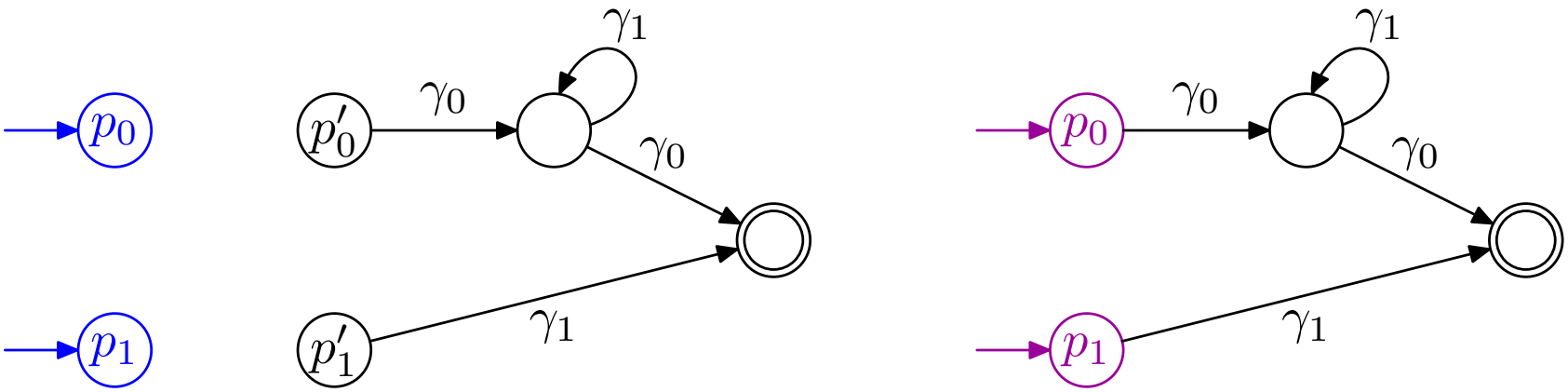

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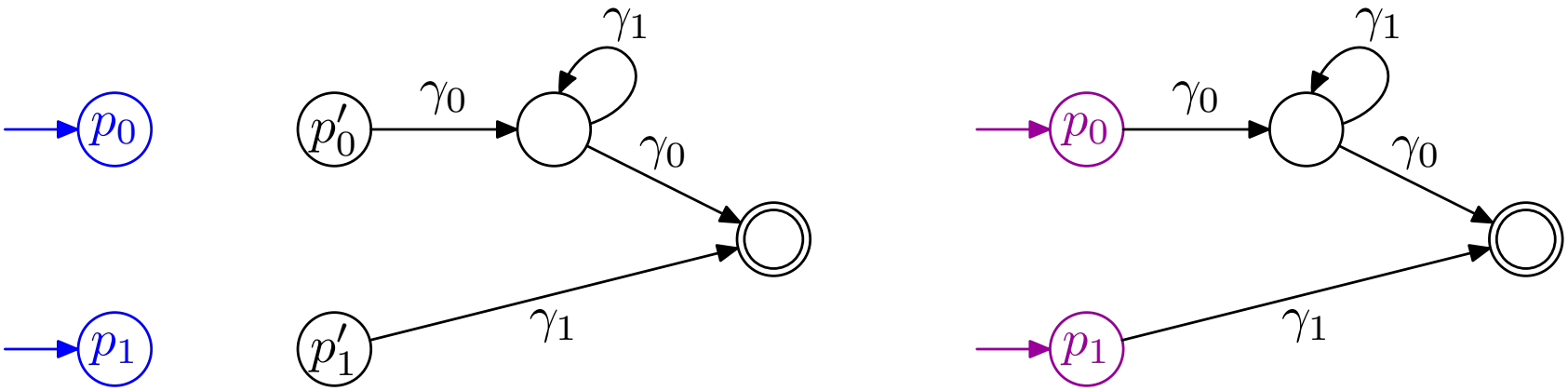

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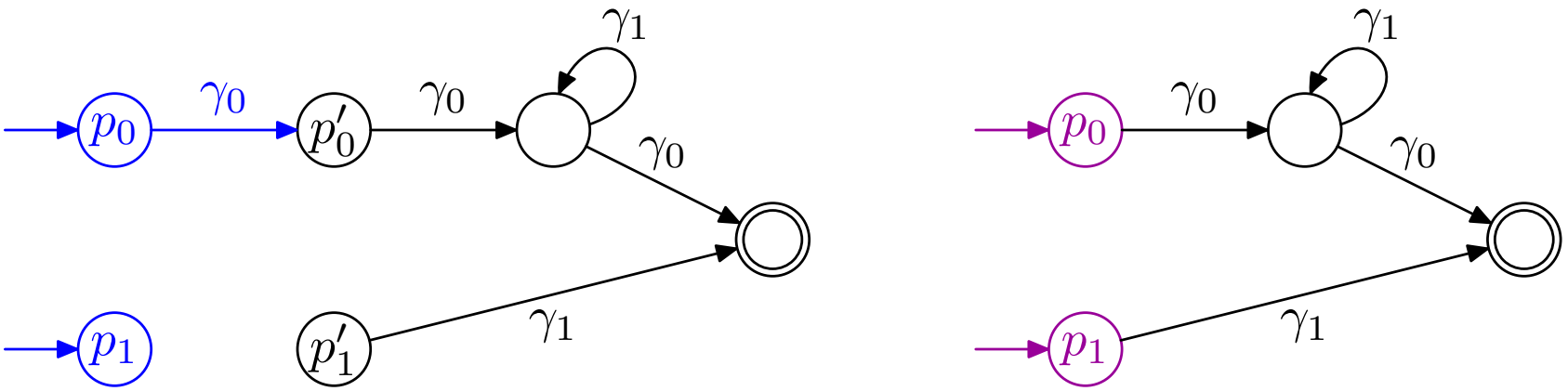

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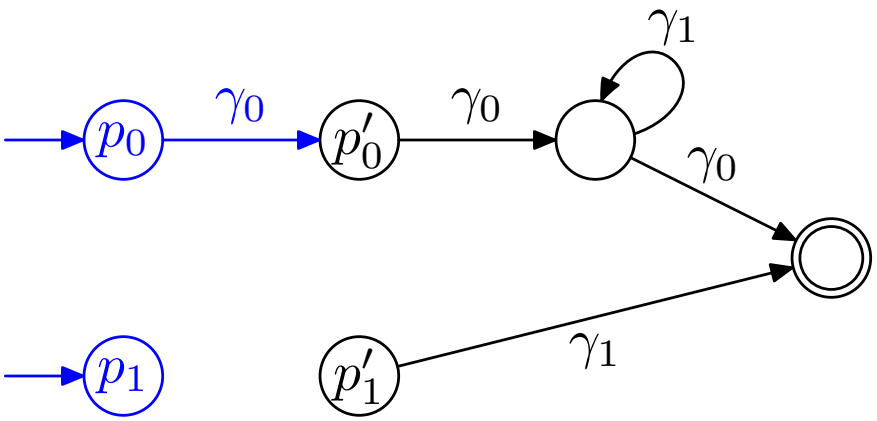

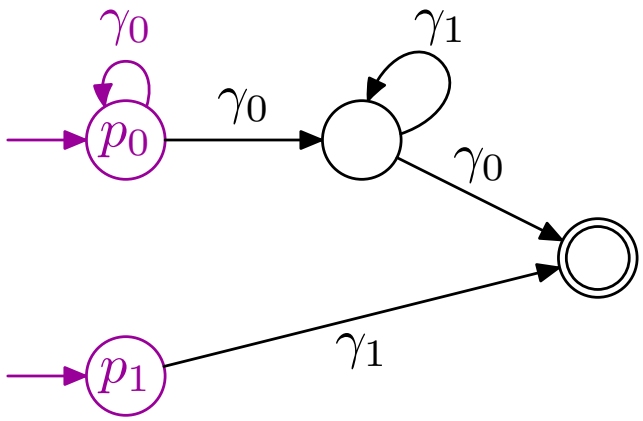

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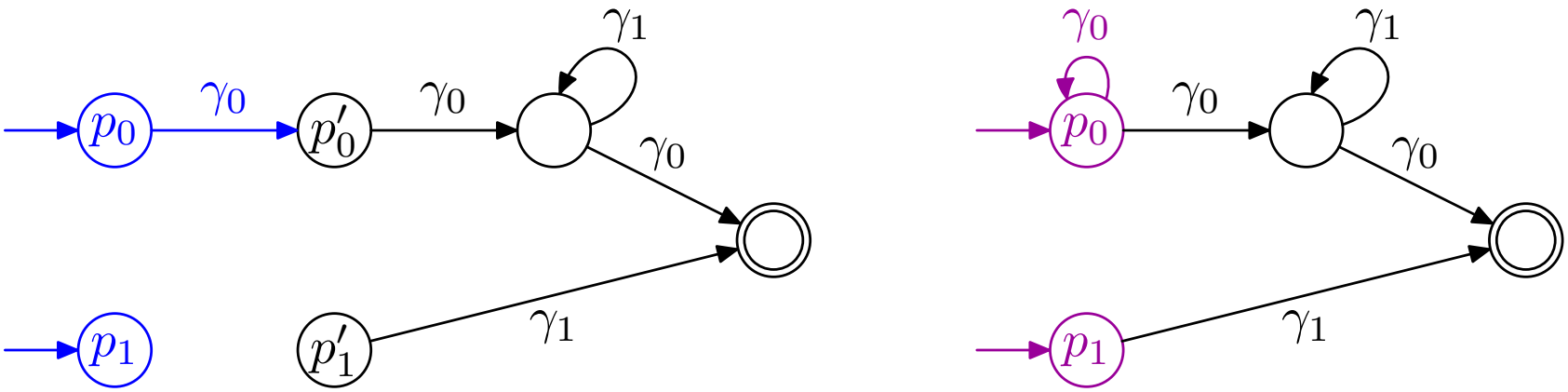

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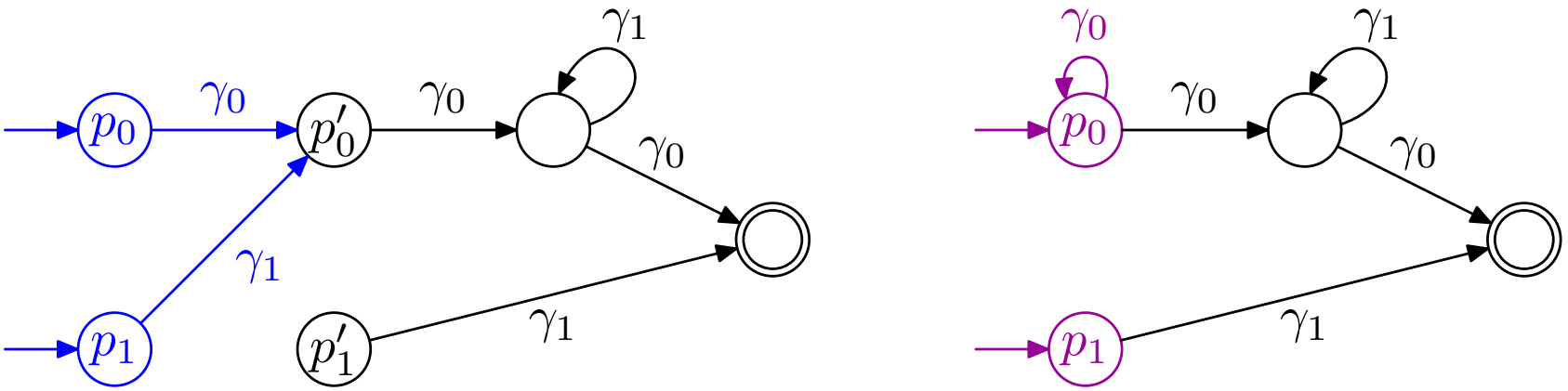

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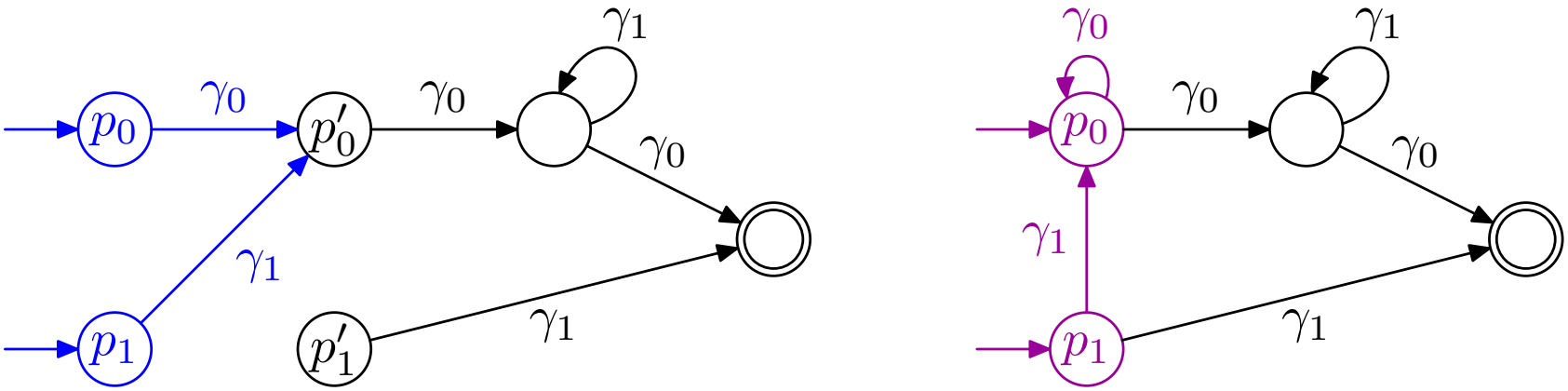

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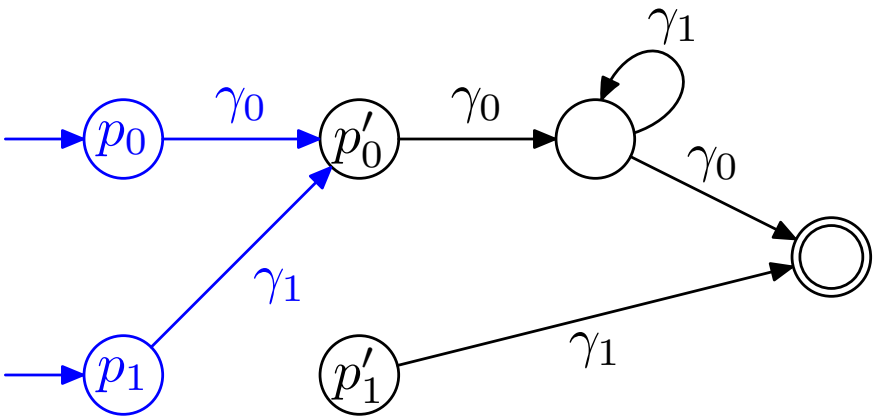

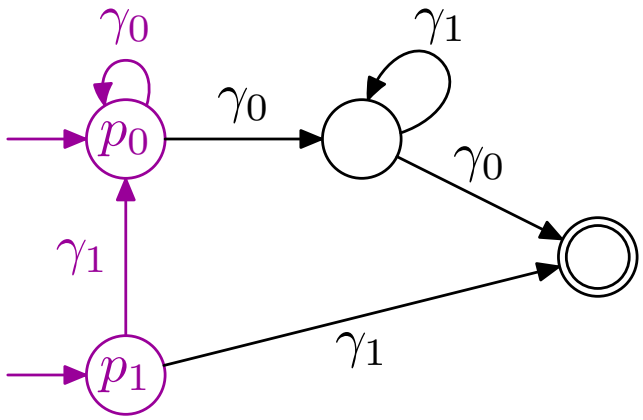

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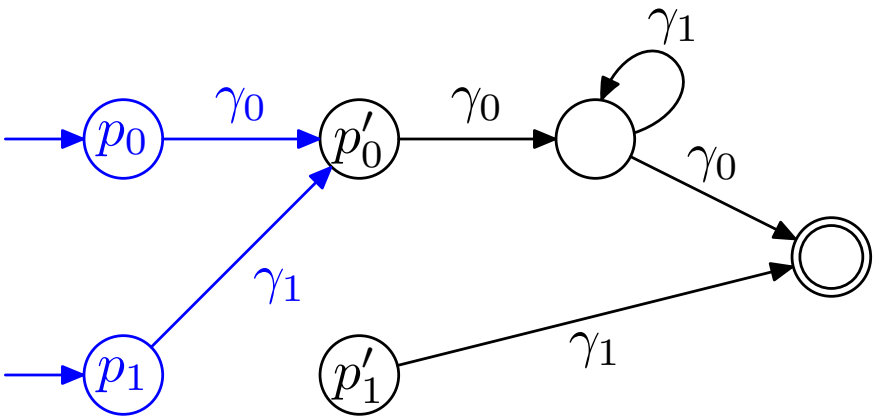

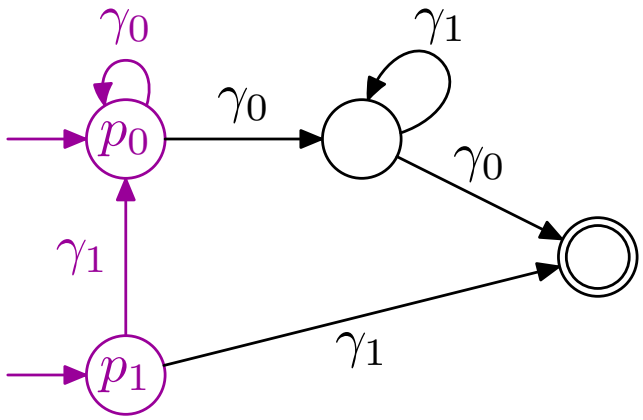

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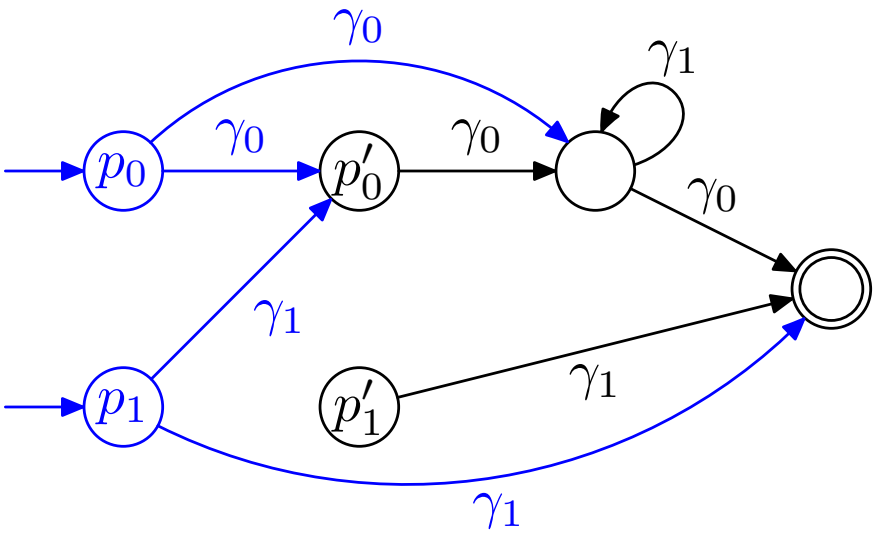

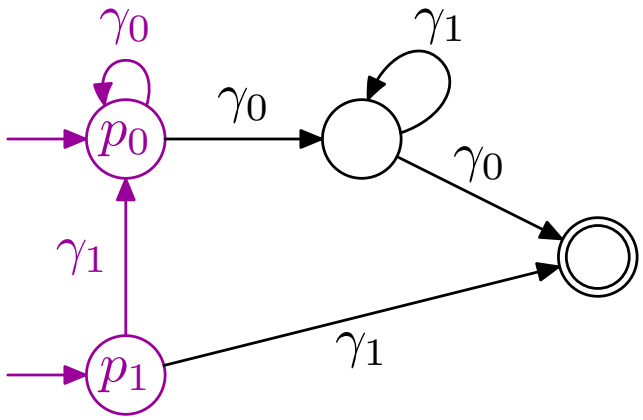

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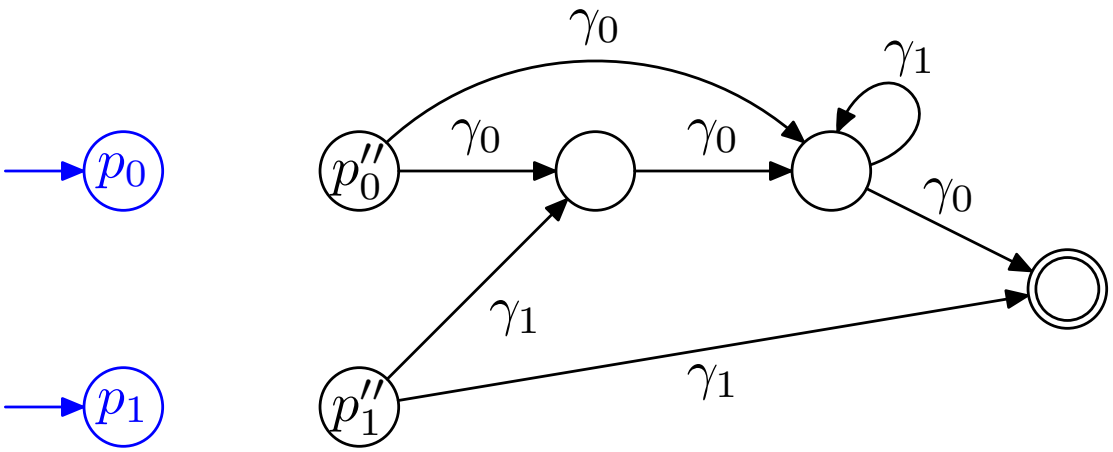

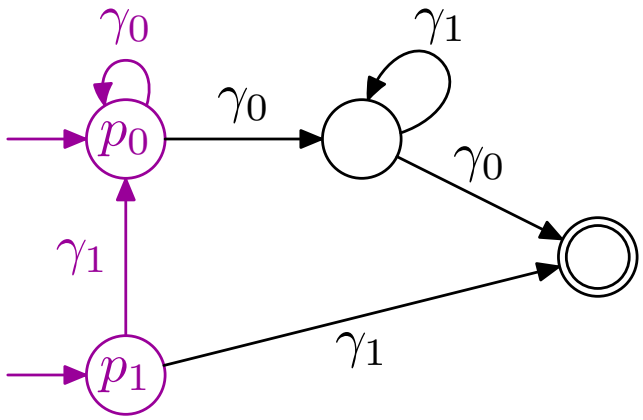

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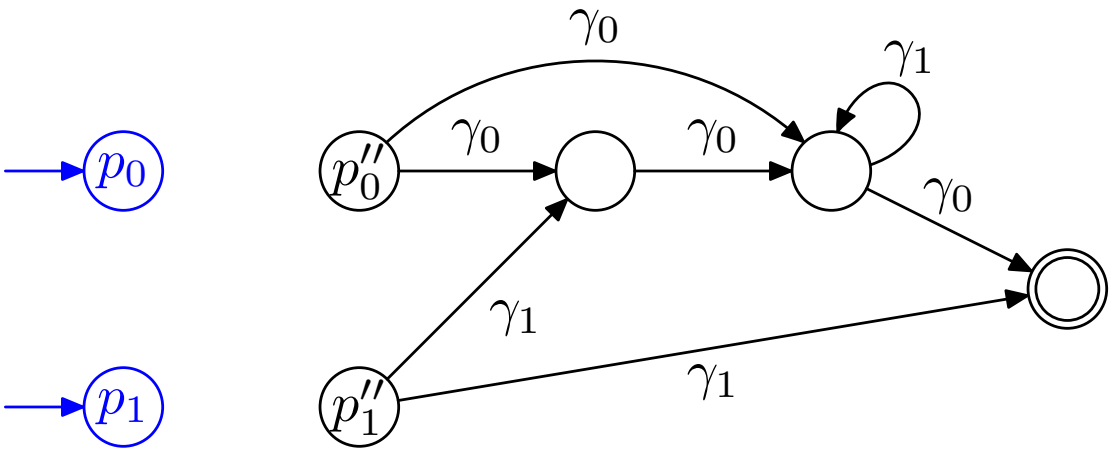

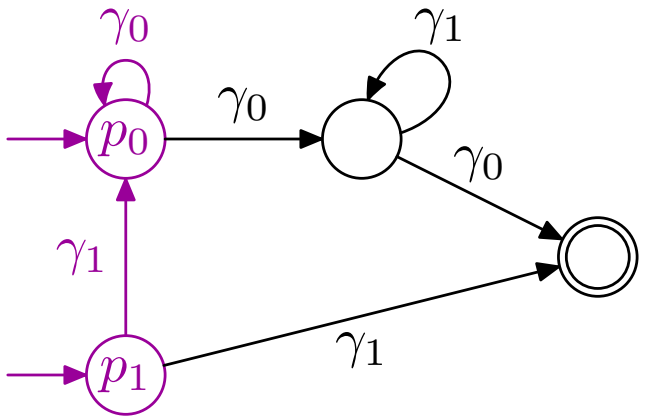

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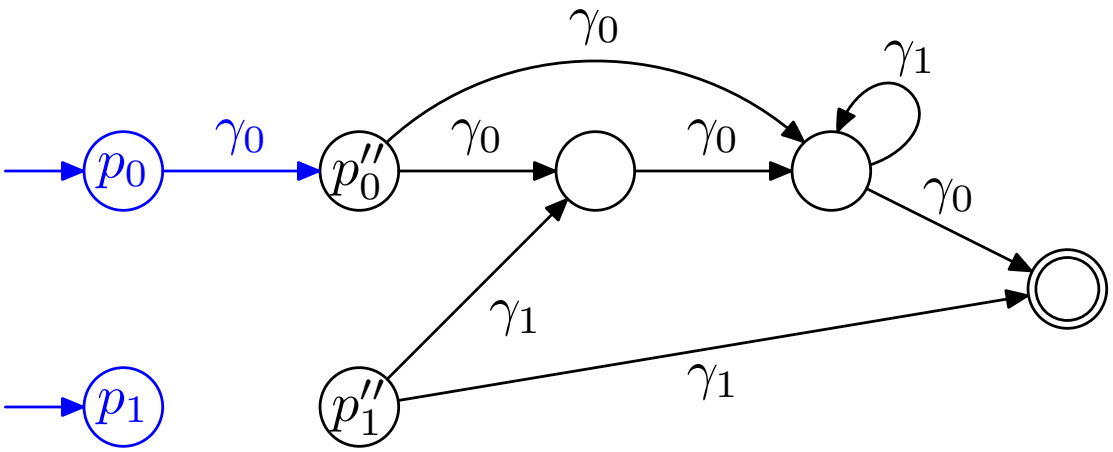

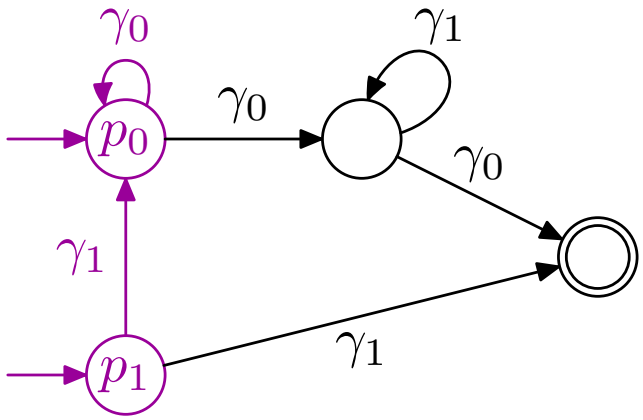

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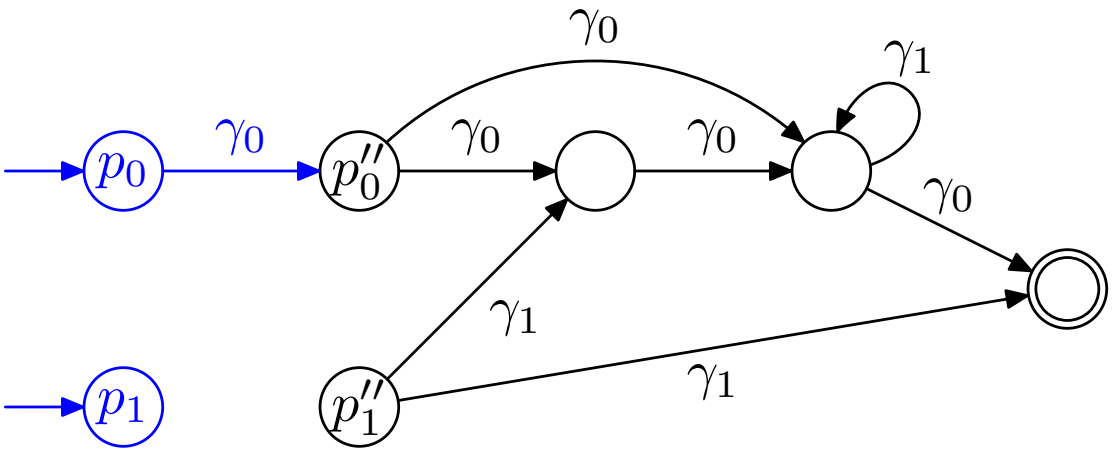

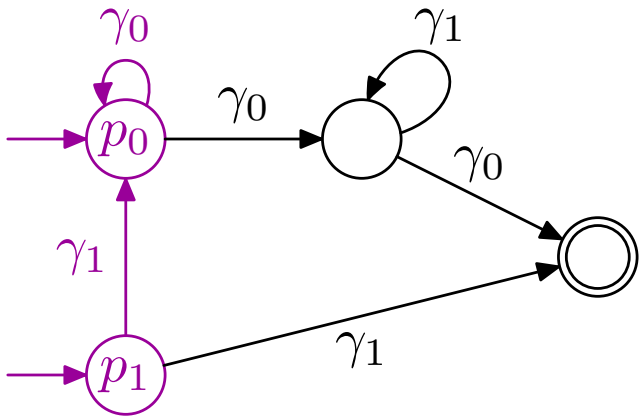

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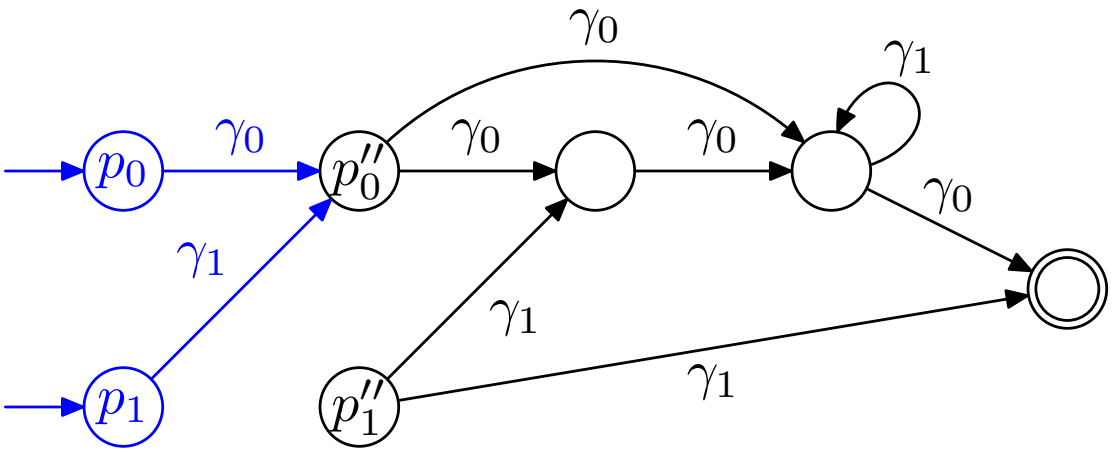

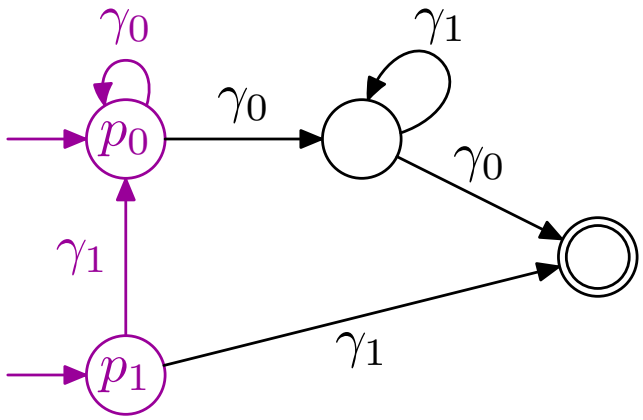

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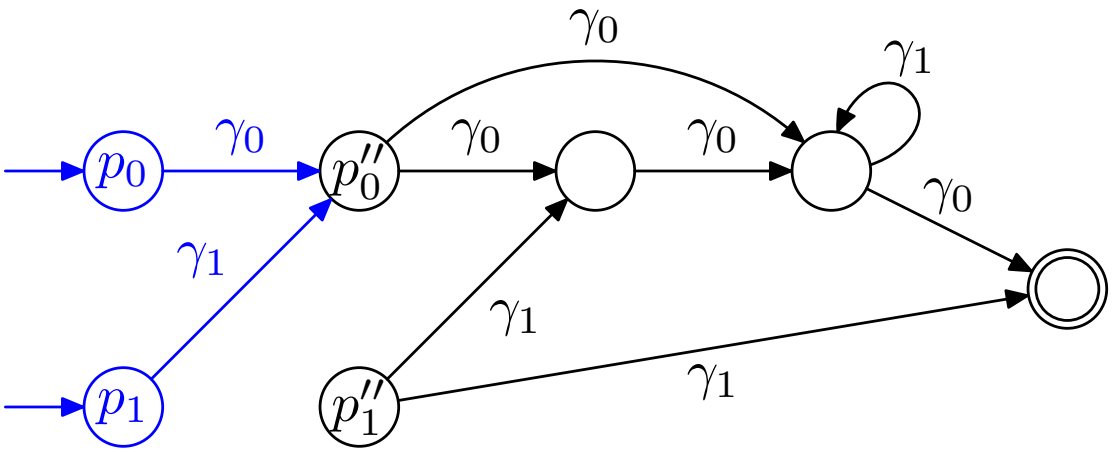

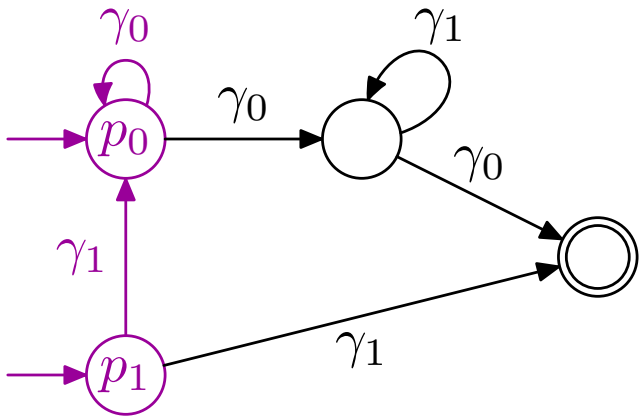

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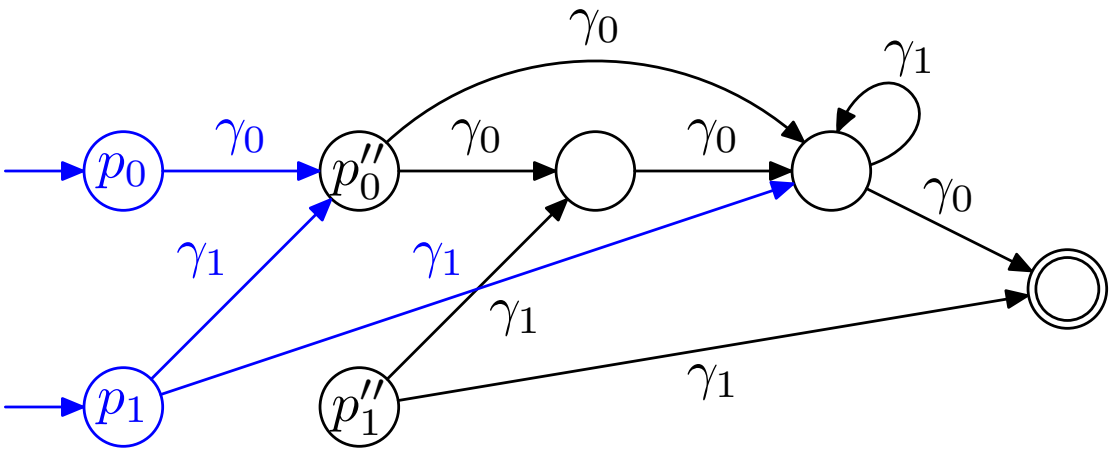

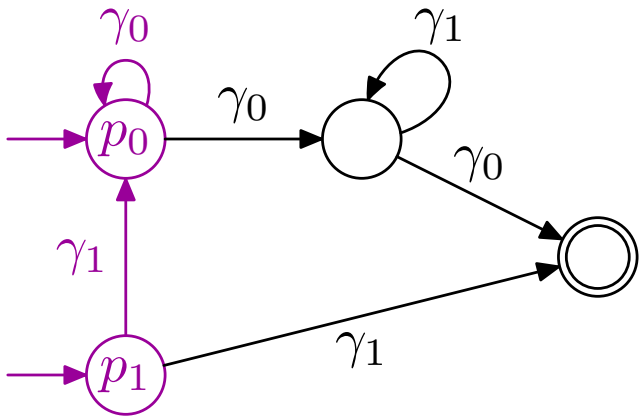

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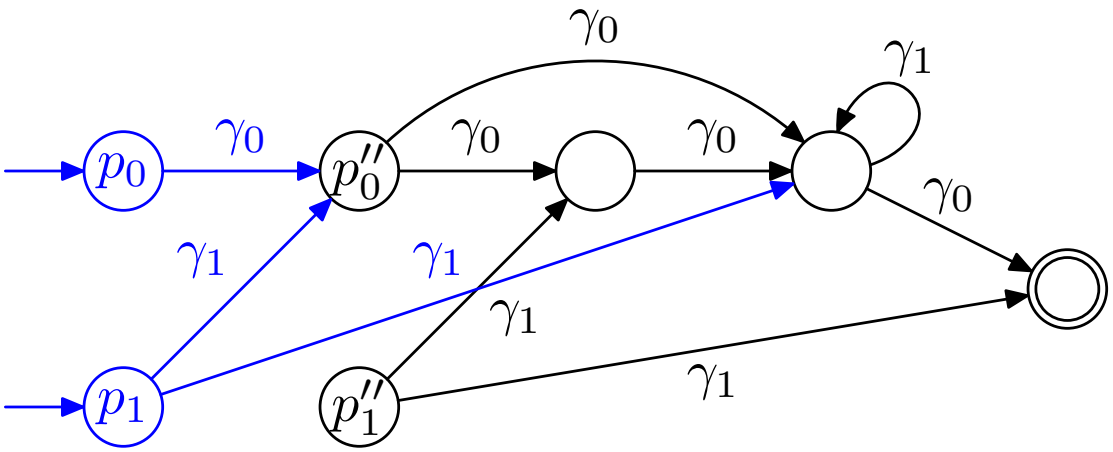

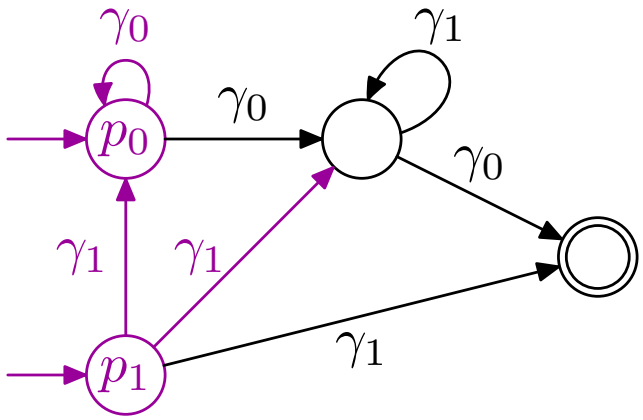

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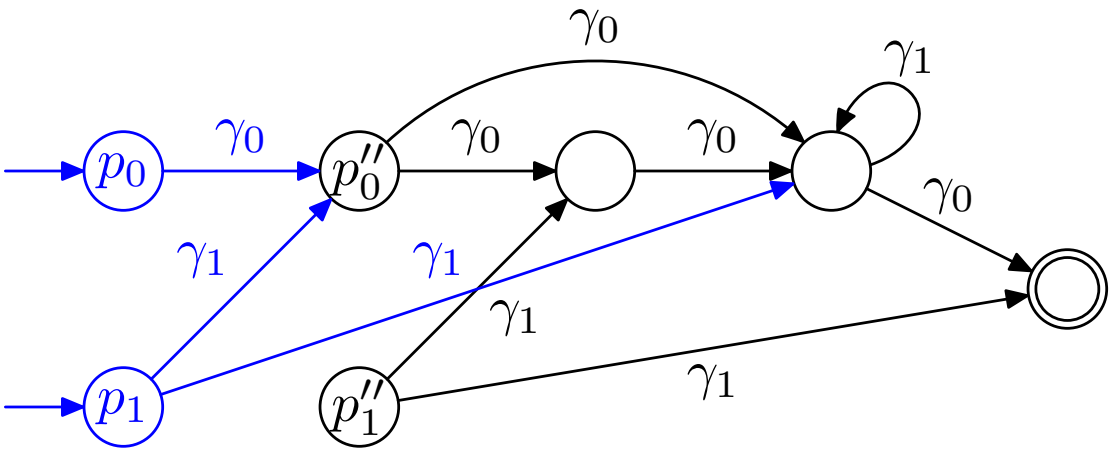

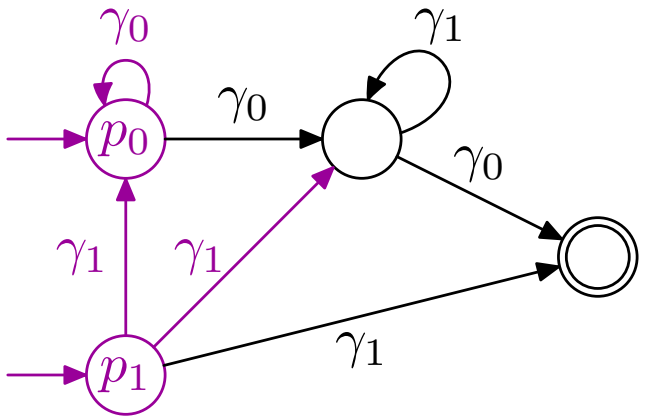

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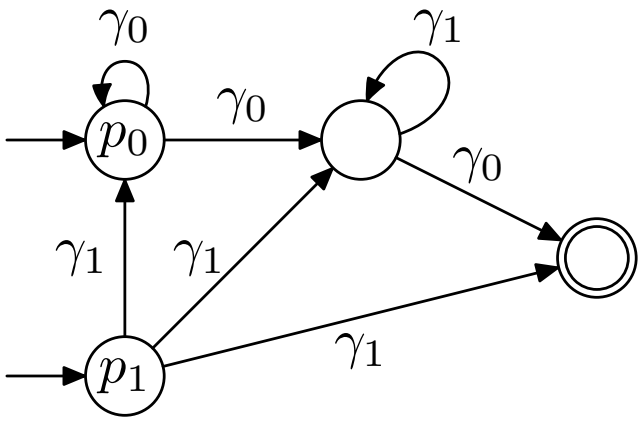

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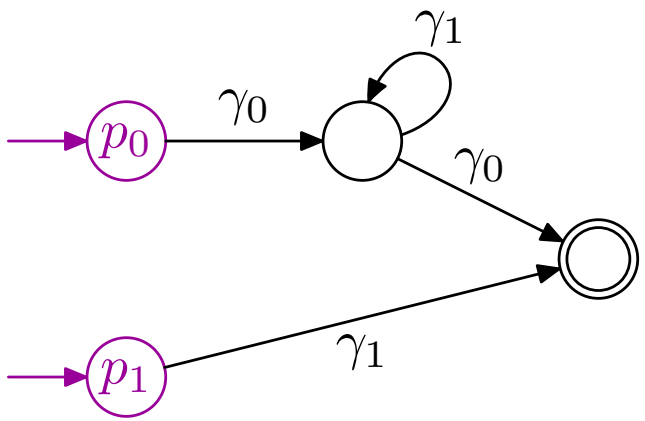

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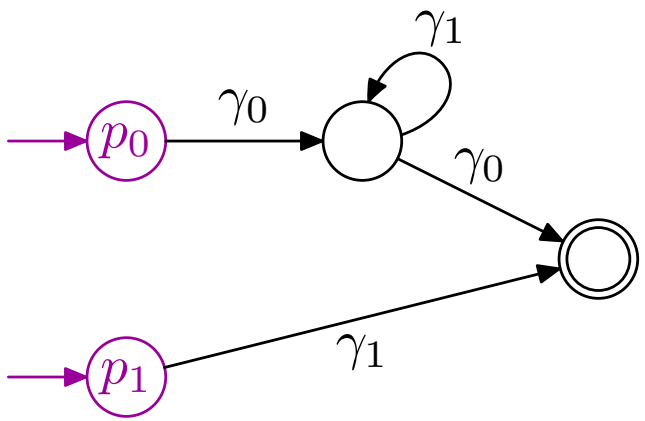

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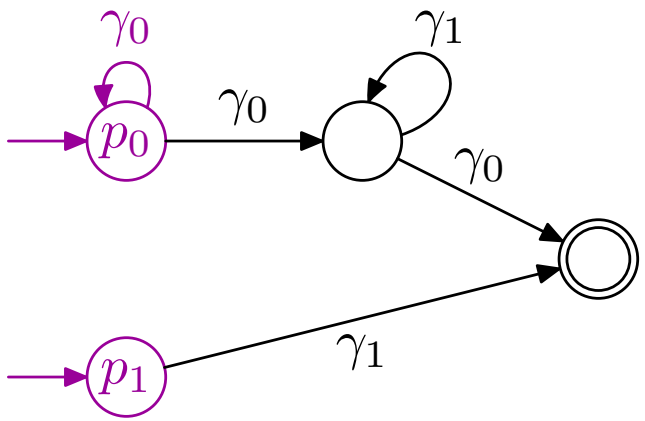

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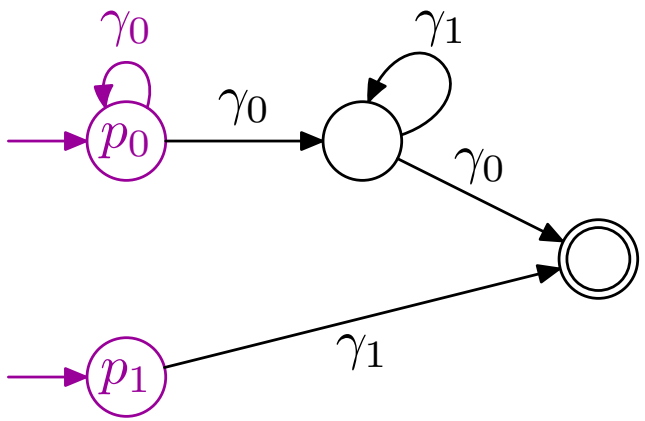

Fortunately: correct if initial states have no incoming arcs.

The proof (1/4)

Input: Pushdown automaton (P, Γ, Δ), NFA $\mathcal{A} = (Q, \Gamma, \rightarrow_0, P, F)$ recognizing a regular set C.

Precondition: No transition of $\mathcal{A}$ leads to an initial state.

Output: NFA $\mathcal{A}_{pre^*} = (Q, \Gamma, \rightarrow, P, F)$.

Postcondition: $\mathcal{A}_{pre^*}$ recognizes $pre^*(C)$.

Algorithm: Add new transitions according to the following saturation rule

> If $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle$ and $p' \xrightarrow{w} q$ in the current automaton, add a transition (p, γ, q).

The proof (2/4)

Goal: show that $\mathcal{A}_{pre^*}$ only recognizes words of $pre^*(C)$.

(Showing that it recognizes all words of $pre^*(C)$ is easy.)

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By induction on j. Basis $j = 0$ is easy.

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$$(1) \qquad (4) \qquad (2) \qquad (5)$$

Finally, if q initial then $w' = \epsilon$ because of (6) and precondition.

Forward search and complexity

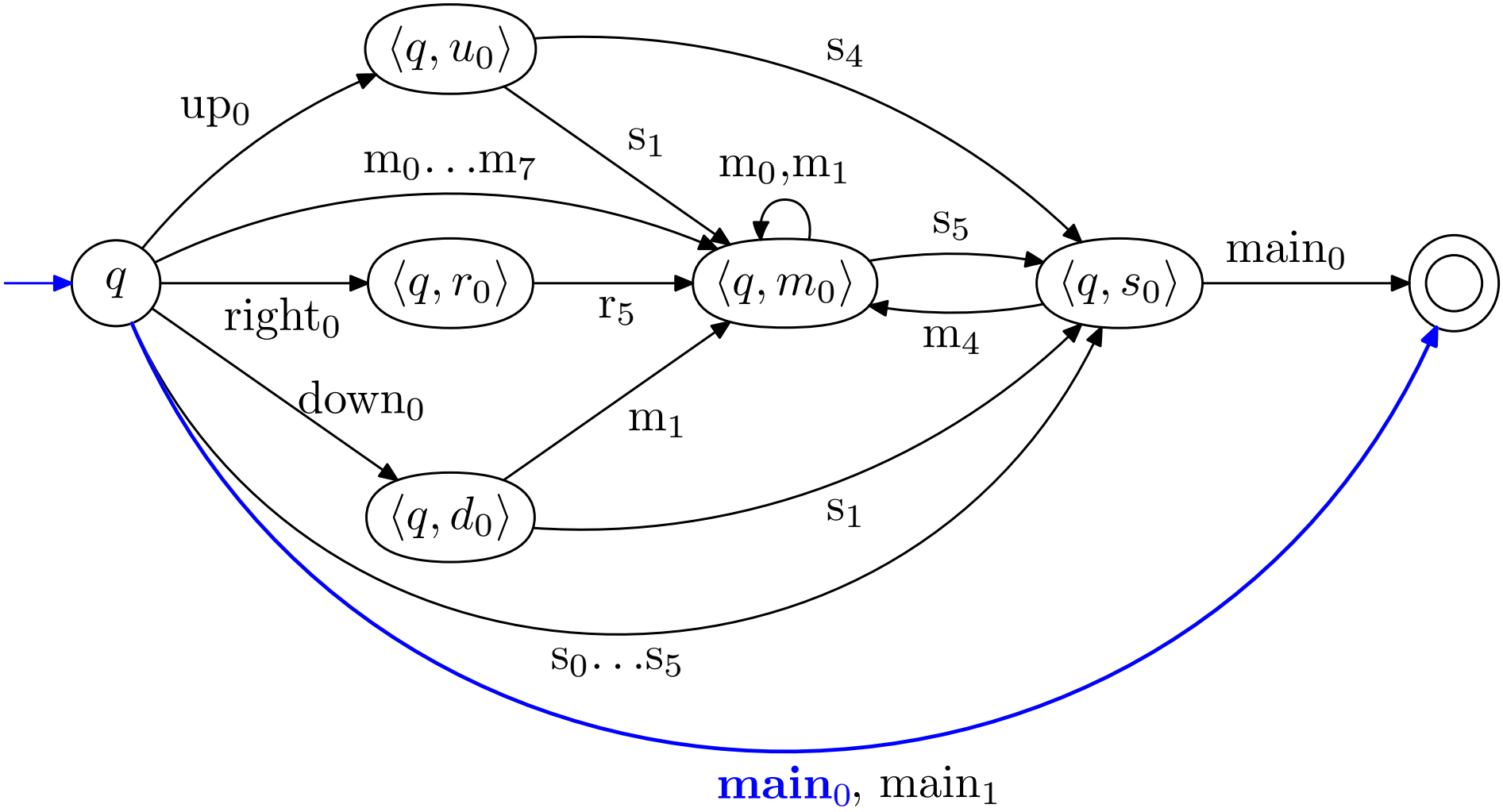
Symbolic forward search with regular sets can be accelerated in a similar way

Recall input: Pushdown automaton (P, Γ, Δ), NFA $\mathcal{A} = (Q, \Gamma, \rightarrow_0, P, F)$.

Complexity of backward search: $O(|Q|^2 \cdot |\Delta|)$ time, $O(|Q| \cdot |\Delta| + | \rightarrow_0 |)$ space.

Complexity of forward search: $O(|P| \cdot |\Delta| \cdot (|Q \setminus P| + |\Delta|) + |P| \cdot | \rightarrow_0 |)$ time and space.

Reachable configurations of the plotter program



Repeated reachability for pushdown systems

Let $I = \langle p_0, \gamma_0 \rangle$ and $D = \langle p, \Gamma^* \rangle$.

D can be repeatedly reached from I iff

$$\langle p_0, \gamma_0 \rangle \longrightarrow^* \langle p', \gamma w \rangle$$

and

$$\langle p', \gamma \rangle \longrightarrow^* \langle p, v \rangle \longrightarrow^* \langle p', \gamma u \rangle$$

for some p', γ, w, v, u.

Repeated reachability can be reduced to computing several pre^*.

To know more

Pushdown automata usually called pushdown processes in our context.

They are equivalent to recursive state machines.

The class of one-state PDAs is interesting, usually studied under the name Basic Process Algebra(BPA) or context-free processes

Some people: Alur, Baeten, Bouajjani, Caucal, E., Etessami, Schwoon, Steffen, Stirling, Yannakakis, Walukiewicz . . .

Tools: Moped, available online at
http://www.informatik.uni-stuttgart.de/fmi/szs/tools/moped/

Technology transfer: the Static Driver Verifier (Microsoft)
see http://www.microsoft.com/whdc/devtools/tools/sdv.mspx

(Lossy) Channel Systems

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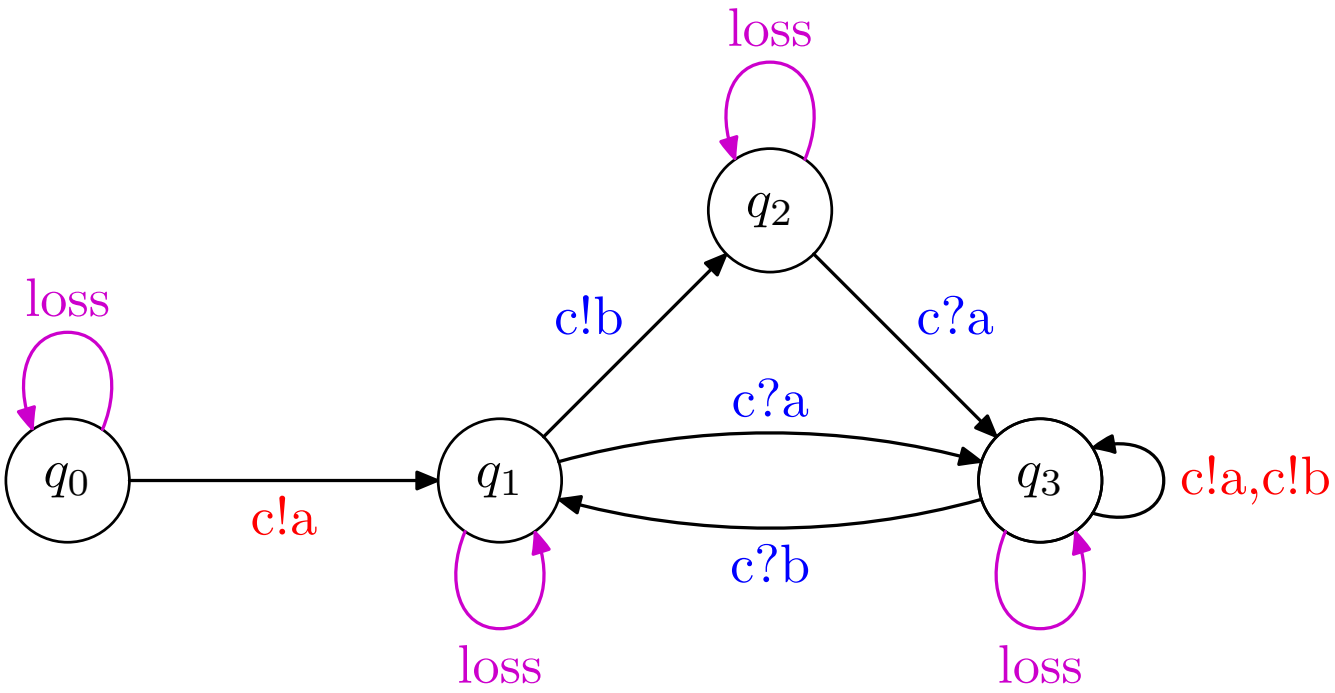

Automata extended with channels (unbounded queues)

Send transitions: no guard, action sends message to the channel.

Receive transitions: guard checks if the channel is nonempty, action removes the first message.

Loss transitions: self-loops, no guard, action removes an arbitrary message.

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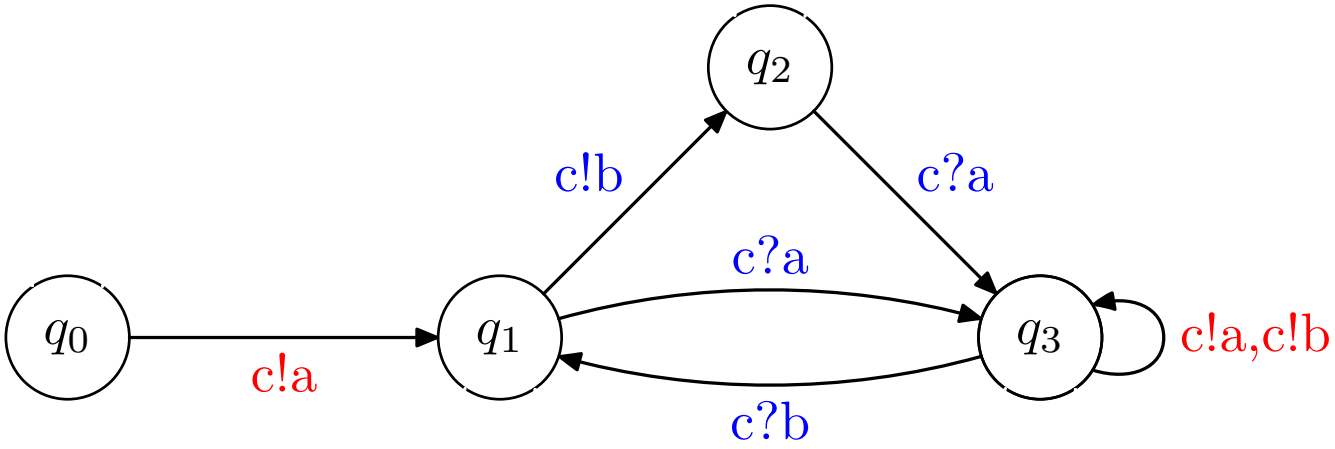

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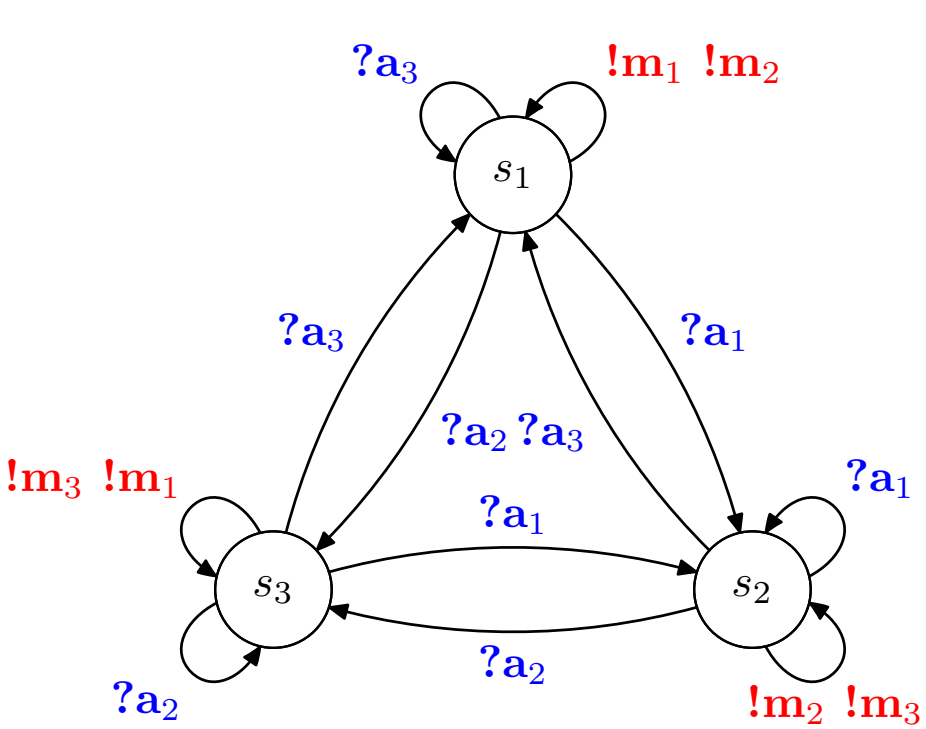
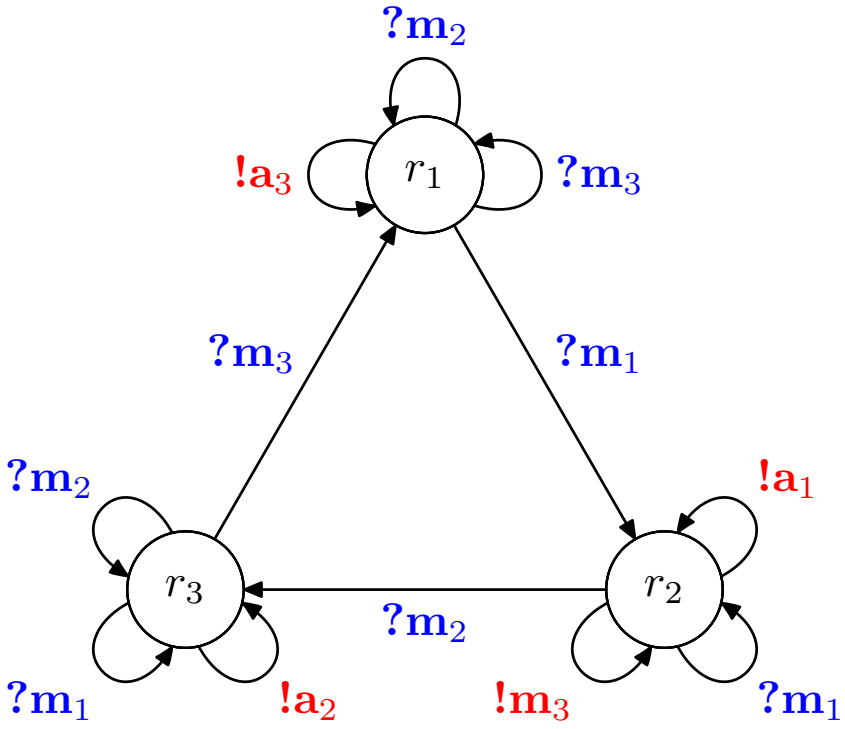
Case study: A sliding window protocol





Symbolic reachability for (lossy) channel systems

Perfect channels: Turing powerful model, even with only one channel.

Lossy channels:

- Backward search: decidable for D upward-closed set
- Forward search: (assume one channel for simplicity)
  Choose $\mathcal{C}$ as the sets of the form $Q \times SRL$, where SRL are the simple regular languages defined as those represented by simple regular expressions (SREs).

$$\begin{aligned} \text{Atomic expression:} &\quad \emptyset \mid \epsilon \mid (a + \epsilon) \mid (a_1 + \ldots + a_m)^* \\ \text{Product:} &\quad e_1 e_2 \ldots e_n \\ \text{SRE:} &\quad p_1 + \ldots + p_n \end{aligned}$$

We show that this choice and $I \in SRL$ satisfies conditions (1)-(5) but not (6).
The fixed point is of the form $Q \times SRL$, but it cannot be effectively computed, and so no 'perfect' acceleration can exist.

Downward-closed sets and regularity

Theorem: Let Σ be a finite alphabet and let $L \subseteq \Sigma^*$. If L is upward- or downward-closed w.r.t. the subword order, then L is regular.

Proof: Let $L \subseteq \Sigma^*$ upward closed. By Higman's lemma, L has finitely many minimal elements $m_1, \ldots, m_k$, and so $L = m_1 \Uparrow + \ldots + m_k \Uparrow$, where $m_i \Uparrow$ denotes the upward-closure of m_i. Since each of the $m_i \Uparrow$ is regular (exercise), we are done.
If L is downward-closed, then $\overline{L}$ is upward-closed. Use now that regular languages are closed under complement.

Theorem: Let Σ be a finite alphabet and let $L \subseteq \Sigma^*$. If L is downward-closed, then $L = L(r)$ for some SRE r.

Proof: Let L be downward-closed and let r be a RE with $L = L(r)$. We find an SRE r' with $L(r) = L(r')$.
Let a DRE be a regular expression with syntax:
$\emptyset \mid \epsilon \mid (a + \epsilon) \mid r_1 + r_2 \mid r_1 r_2 \mid r^*$. If $L(r)$ is downward closed, then there is a DRE $\tilde{r}$ with $L(r) = L(\tilde{r})$. So assume w.l.o.g. that r is DRE. We proceed by induction on the structure of r.

- $r = \emptyset, \epsilon, (a + \epsilon)$. Trivial.
- $r = r_1 + r_2$, $r = r_1 r_2$. Since $L(r_1)$ and $L(r_2)$ are downward-closed, the result follows easily from the induction hypothesis.
- $r = r_1^*$. By induction there is an SRE r_1' with $L(r_1') = L(r_1)$. Let

$$r_1' = e_1 \ldots e_{n_1} + e_{n_1+1} \ldots e_{n_2} + \ldots + e_{n_{(k-1)}+1} \ldots e_{n_k}$$

  Then we have $L = (e_1 + \ldots + e_{n_k})^*$.

Corollary: $post^*(I)$ is of the form $Q \times SRL$ for every lossy channel system and every set I of initial configurations.

Proof: Obvious, because $post^*(I)$ downward closed.

Theorem: The problem of, given $q \in Q$ and $r \in SRE$, decide if $post^*(I) \cap \langle q, \Sigma^* \rangle = \langle q, L(r) \rangle$ is undecidable.

Proof: By reduction from the recurrent state problem: given a lossy channel system and a state q, does some computation visit q infinitely often?

Lemma [Condition (3)]: If $C \in \mathcal{C}$, then $C \cup post(C) \in \mathcal{C}$.

Proof: Easy.

Acceleration through loops

Compute a symbolic reachability graph with elements of $\mathcal{C}$ as nodes:

- Add I as first node
- For each node C and each transition t, add an edge $C \xrightarrow{t} post[t](C)$

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Replace $C \xrightarrow{\sigma} post[\sigma](C)$ by $C \xrightarrow{\sigma} X$, where X satisfies

- $post[\sigma](C) \subseteq X$, and
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Acceleration: given a loop $C \xrightarrow{\sigma} post[\sigma](C)$, replace $post[\sigma](C)$ by

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Question: find a suitable class of loops such that $post[\sigma^*](C)$ belongs to $\mathcal{C}$.

An acceleration for lossy channel systems

Theorem [Abdulla, Bouajjani, Jonsson, CAV'98]: For any loop σ of a lossy channel system and any SRE r, the set $post[\sigma^*](r)$ is an SRE that can be computed in quadratic time in the size of r.

Proof sketch. Show that for every product p and every sequence σ of operations (send or receive) there is a natural number n, linear in the size of p, such that either

(1) $post[\sigma](p) = \emptyset$, or

(2) there is a product p', computable in quadratic time, such that $L(p') = \sum_{j \geq n} post[\sigma^j](p)$.

If (1), then $post[\sigma^*](p) = \sum_{j<n} post[\sigma^j](p)$.
If (2), then $post[\sigma^*](p) = \sum_{j<n} post[\sigma^j](p) \cup L(p')$.

Examples

(1) $\sigma = ?a\,!b\,?c \qquad r = (a+b)^*(b+c)^*(a+c)^*(b+d)^*.$

$n = 2 \qquad p' = (a+c)^*(b+d)^*b^*$

Intuition: after consuming the words in $(a+b)^*(b+c)^*$ the loop can be exectued an arbitrary number of times producing and adding to the right an arbitrary number of bs. The global effect is obtained by concatenating to the right of $(a+b)^*(b+d)^*$ the string b^*.

(2) $\sigma = ?a\,?b\,!a\,!b\,!c \qquad r = (a+c)^*(b+c)^*.$

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Pray for termination.

Channel contents of the sliding window protocol

States	Mess. channel	Ack. channel
s_1, r_1	$(m_2 + m_3)^*(m_1 + m_3)^*(m_1 + m_2)^*$	a_3^*
s_1, r_2	$(m_1 + m_3)^*(m_1 + m_2)^*$	$a_3^* a_1^*$
s_1, r_3	$(m_1 + m_2)^*$	$a_3^* a_1^* a_2^*$
s_2, r_1	$(m_2 + m_3)^*$	$a_1^* a_2^* a_3^*$
s_2, r_2	$(m_1 + m_3)^*(m_1 + m_2)^*(m_2 + m_3)^*$	a_1^*
s_2, r_3	$(m_1 + m_2)^*(m_2 + m_3)^*$	$a_1^* a_2^*$
s_3, r_1	$(m_2 + m_3)^*(m_1 + m_3)^*$	$a_1^* a_2^*$
s_3, r_2	$(m_1 + m_3)^*$	$a_2^* a_3^* a_1^*$
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Recent alternative [Vardhan, Sen, Viswanathan, Agha, FSTTCS '04]:
apply learning algorithms for regular languages.

Angluin's learning setting [I&C '87]

Two agents, the Teacher and the Learner.

The Teacher knows a regular language $L \subseteq \Sigma^*$.

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> (a hypothesis), and asks if $L = H$.
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Question: give an algorithm (a strategy) for the Learner.

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Completeness: the Learner eventually produces L as hypothesis.

Complexity: polynomial in the size of the minimal DFA for L.

First idea: separating sequences

Fix DFA $A = (\Sigma, Q, \delta, q_0, F)$ to be learnt.

A sequence $\sigma \in \Sigma^*$ separates two states $q_1, q_2 \in Q$ if $\delta(q_1, \sigma) \in F$ and $\delta(q_2, \sigma) \notin F$ or vice versa.

A set $T \subseteq \Sigma^*$ is a separating set if every pair of distinct states is separated by some sequence of T.

Every DFA has a separating set of size $|Q| - 1$ (exercise).

The signature of a state q is the function $s_q \colon T \to \{0, 1\}$ given by:

$$s_q(\sigma) = \begin{cases} 1 & \text{if } \delta(q, \sigma) \in F \\ 0 & \text{otherwise} \end{cases}$$

Fact: different states have different signatures.

A first algorithm

Assume we are given and a separating set T of A. The following algorithm constructs A using membership queries only.

- Initialize a stack of sequences to be analyzed.
- Add λ to the stack.
- While the stack is nonempty
  - Pop a sequence σ, and create the state $\overline{\sigma}$.
  - Mark $\overline{\sigma}$ as final iff $\delta(q_0, \sigma) \in F$.
  - Compute the signature of $\delta(q_0, \sigma)$.
  - If the signature has been encountered before in state $\overline{\tau}$, merge the states $\overline{\sigma}$ and $\overline{\tau}$.
  - Otherwise, for every $a \in \Sigma$, push σa onto the stack.

Idea of Angluin's algorithm

Check using equivalence queries that A has at least two states.

Take $T := \{\lambda\}$ as initial candidate for a separating set of A .

Iteratively:

> Use membership queries to construct the automaton corresponding to T, and submit it to the Teacher as hypothesis.
>
> If the hypothesis is correct, terminate. Otherwise, use the counterexample to add a sequence to T separating two states that were not separeted before.

How do we get the new separating sequence?

Finding the separating sequence

Let $H = (\Sigma, Q_H, \delta_H, q_{0H}, F_H)$ be the hypothesis.

Recall: a state of H is of the form $\overline{\sigma}$, where σ is the sequence through which the state was "discovered".

Given $\overline{\sigma} \in Q_H$, we have: $\delta(q_0, \sigma) \in F$ iff $\delta_H(q_{0H}, \sigma) \in F_H$.

Given $\tau \in \Sigma^*$, there is a sequence σ with $\delta_H(q_{0H}, \tau) = \overline{\sigma}$.
We denote this sequence by $[\tau]$.

Observe that for every $\tau \in \Sigma^*$: $\delta(q_0, [\tau]) \in F$ iff $\delta(q_{0H}, [\tau]) \in F_H$.
Because if $\delta_H(q_{0H}, \tau) = \overline{\sigma}$ then $\delta(q_0, [\tau]) \in F$ iff $\delta(q_0, \sigma) \in F$ iff $\delta_H(q_{0H}, \sigma) \in F_H$ iff $\delta_H(q_{0H}, [\tau]) \in F_H$.

Let w be the counterexample distinguishing A and H. Assume w.l.o.g. $\delta(q_0, w) \in F$ and $\delta_H(q_{0H}, w) \notin F_H$.

Then we have:

$$\delta(q_0, w) \in F \qquad \text{but} \qquad \delta(q_0, [w]) \notin F$$

Assume $w = a_1 a_2 \dots a_n$.

Let i be the smallest index with

$$\delta(q_0, [a_1 \dots a_i] a_{i+1} \dots a_n) \in F \qquad \text{but} \qquad \delta(q_0, [a_1 \dots a_{i+1}] a_{i+2} \dots a_n) \notin F$$

Then the sequence $a_{i+2} \dots a_n$ separates the states of A reached by the sequences $[a_1 \dots a_i] a_{i+1}$ and $[a_1 \dots a_{i+1}]$.

Notice that these states were not separated so far by any sequence (why?).

Set $T := T \cup \{a_{i+2} \dots a_n\}$

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> . . . but equivalence queries still hopeless.

Third attempt

Define a marked transition sequence (MTS) as a pair (σ, c), where σ is a sequence of transition names and c is a configuration.
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Replace equivalence queries by containment queries.

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2. $H \supseteq \mathit{Exec}$ and $H \cap D \neq \emptyset$, then the Teacher returns $(\sigma, c) \in H \cap D$.
   The Learner checks whether $(\sigma, c) \in \mathit{Exec}$:

   2.1. if $(\sigma, c) \in \mathit{Exec}$, then the Learner has learned a DE, stop;

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. . . but checking $H \supseteq \mathit{Exec}$ is also hopeless!

Pre-fixpoint queries (1/3)

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The clever idea:

If $c \xrightarrow{t} c'$, then say $(\sigma, c) \rightarrow (\sigma t, c')$. Given a set M of MTSs, let

$$\mathit{post}(M) = \{m \mid \exists m' \in M \wedge m' \rightarrow m\}$$

Exec is the least fixed point of the equation $X = \mathcal{F}(X)$ where

$$\mathcal{F}(X) =_{def} \{(\epsilon, c) \mid c \in I\} \cup X \cup \mathit{post}(X)$$

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We replace the query $H \supseteq \mathit{Exec}$ by the query $\mathcal{F}(H) \subseteq H$.

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Algorithms for the remaining problems follow easily from the Corollary.

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The assumption '*Exec* is regular' may depend on the encoding use to represent a pair (σ, c) as a word.