The Complexity of Nash Equilibria and Fixed Points of Algebraic Functions.
Homework exercises

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1. This first exercise is very easy, and is just intended to get you started thinking about Nash Equilibria.

   Consider the following n-player normal form game.

   In a class room with 50 students, each student has to pick a whole number from 1 to 10000. The student should then write the number on a piece of paper with their name on it (in secret without revealing it to any other student), and submit this to the teacher. After everyone does this, the teacher should open and reads all the numbers.

   The student who picked a number that is closest to half of the average of the numbers picked by all players wins a payoff of 1000 Zloty. All other players get a payoff of 0. If there are ties for who is closest to half the average, the payoff of 1000 is split equally among those who are closest to half the average. (So if m students tied for closest, they each get payoff $1000/m$, thus we assume one can have fractional payoffs in Zloty.).

   (a) Assume you are a student. What would your strategy be in such a game?

   (b) What are the Nash equilibria in this game?

   (c) What if we change the rules slightly, so that if there are ties, all students who are tied for closest get payoff 1000 Zloty. What would your strategy be? What are the Nash equilibria?

2. This next exercise is similarly very easy, and has a lot of similarities to the previous one, but is perhaps more surprising and eye-opening.

Consider the following 2-player game:

The game consists of at most 100 rounds, after which the game ends.

Each player starts with 1 Zloty.

Initially, we are at round 1. Each round proceeds as follows:

(a) Player 1 can choose to either give player 2 one Zloty or not.

If player 1 chooses not to give one Zloty, the game ends immediately, and both players keep the money they currently have.

Otherwise, if player 1 chooses to give player 2 one Zloty, then then GOD matches this and also gives player 2 one extra Zloty (so in total, player 2 gets two Zloty as a result of this).

(b) Then it is player 2's turn to choose whether to do exactly the same thing: either give player 1 a Zloty or not.

If player 2 chooses not to give the Zloty, again, the game ends immediately and both players keep the money they have.

If player 2 chooses to give the Zloty , then GOD matches this again, giving player 1 an extra Zloty.

The game proceeds like this and ends after 100 rounds (unless the game terminated earlier).

Questions:

(a) What would your strategy be in this game if you were player 1? What if you were player 2?

(b) What are the Nash equilibria of this game?

(c) In light of your answers, discuss what the implications are, if any, for the concept of Nash equilibrium.

3. This question is intended to give you a more quantitative understanding about 2-player Nash Equilibria.

Consider the following 2-player strategic game:

$$\begin{bmatrix} (4,4) & (5,2) & (4,1) \\ (7,1) & (2,6) & (3,4) \end{bmatrix}$$

(a) Find a Nash equilibrium in this game. Say what the expected payoff is for the two different players under this Nash equilibrium.

(b) Are there any other Nash equilibria? Explain your answer.

4. This question asks you to elaborate on something that was mentioned in the lectures:

   Recall the *square-root sum problem*, described in the lectures: we are given as input, I, a tuple of natural numbers $(d_1, \ldots, d_n)$, together with another natural number k, and the problem is to decide whether $\sum_i \sqrt{d_i} \leq k$.

   Recall also, as mentioned in the lectures, that there is a polynomial time algorithm to detect whether $\sum_i \sqrt{d_i} = k$, so we can assume without loss of generality that either $\sum_i \sqrt{d_i} < k$ or $\sum_i \sqrt{d_i} > k$.

   Suppose there is a polynomial time computable function f, which given an instance I of the square-root sum problem, produces a game $\Gamma_I = f(I)$, with the following properties. computable function f, such that:

   (a) The game Γ_I has a unique Nash equilibrium, x^*.

   (b) If $\sum_i \sqrt{d_i} < k$ then $x^*_{1,1} < 1/2$, and if $\sum_i \sqrt{d_i} > k$, then $x^*_{1,1} > 1/2$.

   (In other words, in the NE, the probability with which player 1 plays its first strategy is either less than or greater than 1/2, depending on whether the square-root sum is greater than k or less than k, respectively.)

   (Indeed such a reduction exists: see our paper.)

   Explain how to derive from this a new polynomial-time computable function, g, such that given an instance, I of the square-root sum problem, the reduction produces a game $\Gamma'_I = g(I)$, which has one more player, than Γ_I (call this extra player, Player 0), and such that the following hold:

   (a) The game Γ'_I has a unique Nash equilibrium, y^*.

   (b) If $\sum_i \sqrt{d_i} < k$ then $y^*_{0,1} = 0$, and if $\sum_i \sqrt{d_i} > k$, then $y^*_{0,1} = 1$.

   (In other words, in the NE, the probability with which player 0 plays its first strategy is either 0 or 1, depending on whether the square-root sum is greater than k or less than k, respectively.)

5. This question (harder than the previous questions) relates to the quantitative behavior of 3 player games.

   Consider the following 3-player game:

   The strategy sets for the three players are:

$S_1 = \{0, 1\}$, $S_2 = \{0, 1, 2\}$, and $S_3 = \{0, 1, 2\}$.

The payoff table is as follows.

First, we describe the payoff function, u_1, for player 1:

For all $s_2 \in S_2$ and $s_3 \in S_3$, $u_1(0, s_2, s_3) = 0$.

For all $s_3 \in S_3$, $u_1(1, 0, s_3) = -2$.

For all $s_3 \in S_3$, $u_1(1, 1, s_3) = 0$.

For all $s_3 \in S_3$, $u_1(1, 2, s_3) = 4$.

Next, we describe the payoff function, u_2, for player 2:

For all $(s_1, s_2, s_3) \in S_1 \times S_2 \times S_3$:

$$u_2(s_1, s_2, s_3) = \begin{cases} 1 & \text{if } s_2 = s_3 \\ 0 & \text{otherwise} \end{cases}$$

Finally, we describe the payoff function, u_3, for player 3:

$$u_3(0, s_2, s_3) = \begin{cases} -1 & \text{if } s_2 = s_3 \\ 1 & \text{if } s_1 + 1 \equiv s_3 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_3(1, s_2, s_3) = \begin{cases} 1 & \text{if } s_3 \neq 0, \text{ and } s_2 + 1 \equiv s_3 \pmod 3 \\ 0 & \text{if } s_3 = 0, \text{ and } s_2 + 1 \equiv s_3 \pmod 3 \\ -1 & \text{if } s_2 = 0, \text{ and } s_2 + 1 \not\equiv s_3 \pmod 3 \end{cases}$$

Prove that this game has a unique Nash Equilibrium, in which player 1 plays strategy 0 with probability $1/\sqrt{2}$.

(EXTRA CREDIT:) Can you show how to modify the payoffs of this game so that instead of this probability being $1/\sqrt{2}$, it is $1/\sqrt{N}$, for any natural number $N > 1$?

6. (EXTRA CREDIT. This problem was posed to me by Damian Nawinski after my lectures, and I don't know a good answer for it. It is probably quite difficult.)

   Recall that in the lectures we stated that computing the value for *simple stochastic games* (SSGs) is in PPAD (see pages 44-45 of the lecture slides, and of course, for proofs see our full paper).

It is known that the *parity game* problem is polynomial-time reducible to deciding whether for a given SSG, the game value is $> 1/2$. (This is usually proved via an intermediate reduction via discounted mean payoff games, but never mind.) Therefore, by combining these results, we know the parity game problem is in PPAD.[1]

This question asks you to do the following: provide a direct proof of the fact that the parity game problem (i.e., deciding which player has a winning strategy in a given parity game) is polynomial-time reducible to the standard complete problem for PPAD, described on page 20 of the lecture slides.

[1]Strictly speaking, the parity game problem is a decision problem, whereas PPAD is a search problem class. So, by saying the problem is in PPAD, we mean that we can reduce the parity game problem to a search problem in PPAD, such that given a solution to the PPAD problem, a P-time query on that solution answers who has a winning strategy in the Parity game. This is a standard notion of polynomial-time search problem reduction.