# All-Pairs shortest paths via fast matrix multiplication

## Practice exercises

0. Show that $(A^k)_{ij} = 1$ iff there is a path of length $k$ form $i$ to $j$.

1. Obtain $O(n^\omega)$-time algorithms for finding, in a directed graph: (a) a triangle; (b) a *simple* quadrangle; (c) a *simple* cycle of length $k$, where $k \geq 3$ is a constant. (Hints: In an acyclic graph all paths are simple. In (c), the running time may be exponential in $k$. Randomization makes the solution much easier.)

2. Let $G = (V, E)$ be an unweighted *undirected* graph. Let $G^2 = (V, E^2)$ be the square of $G$, i.e., $(u, v) \in E^2$ if and only if $(u, v) \in E$ or there exists $w \in V$ such that $(u, w), (w, v) \in E$. For two vertices $u, v \in V$, let $\delta(u, v)$ and $\delta^2(u, v)$, respectively, be the distances between $u$ and $v$ in $G$ and $G^2$. Prove the following claims on which the correctness of Seidel's algorithm is based:

   (a) $\delta^2(u, v) = \lceil \frac{\delta(u,v)}{2} \rceil$, for every $u, v \in V$.

   (b) If $\delta(u, v) = 2\delta^2(u, v)$, the for every neighbor $w$ of $v$ in $G$ we have $\delta^2(u, w) \geq \delta^2(u, v)$.

   (c) If $\delta(u, v) = 2\delta^2(u, v) - 1$, the for every neighbor $w$ of $v$ in $G$ we have $\delta^2(u, w) \leq \delta^2(u, v)$, with a strict inequality for at least one neighbor.

   What goes wrong when the graph is *directed*?

3. (*) Obtain a version of Seidel's algorithm that uses only *Boolean* matrix multiplication. (Hint: Consider distances also modulo 3.)

4. (a) Obtain a deterministic $O(n^\omega)$ time algorithm for finding *witnesses* for all positions in a Boolean matrix multiplication that have *unique* witnesses.

   (b) Let $1 \leq d \leq n$ be an integer. Obtain a *randomized* algorithms for finding witnesses for all positions that have between $d$ and $2d$ witnesses. (Hint: use sampling.)

   (c) Obtain an $O(n^\omega \log n)$ time algorithm for finding witnesses for all positions.

5. Obtain an $O(n^\omega \log n)$-time algorithm for computing the *diameter* of an unweighted directed graph on $n$ vertices.

6. Let $G = (V, E)$ be a weighted directed graph and let $s$ be a number. We say that a subset $B \subseteq V$ is an *$s$-bridging set* if and only if for every two vertices $u, v \in V$, if every shortest path from $u$ to $v$ uses at least $s$ edges there exists $w \in B$ such that $\delta(u, v) = \delta(u, w) + \delta(w, v)$.

   (a) Show that if $B$ is a random subset of $V$ of size $9n \ln n/s$, then $B$ is an $s$-bridging set, with high probability.

   (b) Suppose that in the $i$-th iteration of the APSP algorithm of Zwick for weighted directed graphs, instead of using a random subset of size $9n \ln n/s$, where $s = (3/2)^{i+1}$, we use an $s/3$-bridging set. Show that the modified algorithm works correctly on unweighted graphs, but may fail on weighted graphs.

   (c) Suitably modify the notion of bridging sets so that it could also be used for weighted graphs.