Geometry of Polynomials Problems, MIMUW Minicourse Jan 9-11 2020

1. (a) Prove that if $a + bx_1 + cx_2 + dx_1x_2$ is real stable then $bc \geq ad$.
(b) Prove that if a polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is homogeneous and real stable, then its nonzero coefficients must be either all positive or all negative.

2. (a) Prove the Cauchy-Binet formula: for any m vectors $v_1, \ldots, v_m \in \mathbb{C}^n$:
$$\det\left(\sum_{i=1}^{m} v_i v_i^*\right) = \sum_{S\subset[m]:|S|=n} \det\left(\sum_{i\in S} v_i v_i^*\right).$$
(b) Define the Laplacian of a graph $G = (V, E)$ as $L = \sum_{ij\in E}(e_i - e_j)(e_i - e_j)^*$ where e_i are the standard basis vectors. Show that the kernel of the Laplacian of a connected graph is equal to the subspace of constant vectors.
(c) The *spanning tree polynomial* of G is a multivariate polynomial in variables $\{z_e\}_{e\in E}$ defined as
$$p_T(\mathbf{z}) := \sum_T \prod_{e\in T} z_e,$$
where the sum is over all spanning trees of T. Show that p_T is real stable.

3. Let T be the infinte $d-$ary tree rooted at r. Show that $W_T^\ell(r)$, the number of closed walks of length ℓ rooted at r in T, satisfies
$$W_T^\ell(r) = (2\sqrt{d-1} + o_\ell(1))^\ell,$$
where $o_\ell(1) \to 0$ as $\ell \to \infty$. (this fact was claimed in Lecture 3)

4. (a) Show that the matching polynomial of a tree is equal to the characteristic polynomial of its adjacency matrix. (b) Let G be a graph, let v be any vertex in G, and let $T_G(v)$ be the self-avoiding walk tree of G rooted at v. Show that the matching polynomial $\mu_G(x)$ *divides* (as a polynomial) the characteristic polynomial of the adjacency matrix of $T_G(v)$. This implies that the roots of the matching polynomial are a subset of the eigenvalues of the adjacency matrix of any self-avoiding walk tree of G, strengthening the upper bound we proved in Lecture 3. (hint: consider the vertex deletion recurrence and rational function which appeared in lecture).

5. *Generalized Gurvits Theorem.* (a) Suppose $g(x) = c_0 + c_1x + \ldots + c_dx^d$ is real rooted with nonnegative coefficients. Fix an integer $k \leq d$ and let
$$\alpha := \inf_{x>0} \frac{g(x)}{x^k}.$$
Show that
$$c_k \leq \alpha \leq f(k) \cdot c_k$$
where
$$f(k) := \left(\inf_{z>0} \frac{e^{cz}}{z^k}\right) / \frac{c^k}{k!},$$
for any constant $c \geq 0$ (show that $f(k)$ depends only on k but not on c). (b) Suppose $g(x_1, \ldots, x_n)$ is real stable with nonnegative coefficients, and let β be the coefficient of $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$. Show by induction that
$$\beta \leq \inf_{x_1,x_2,\ldots,x_n>0} \frac{g(x_1, \ldots, x_n)}{x_1^{k_1}\ldots x_n^{k_n}} \leq \beta \prod_{i=1}^{n} f(k_i).$$