Dynamic Graph Algorithms

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University of Copenhagen

November 14, 2019

Graph algorithms and data structures

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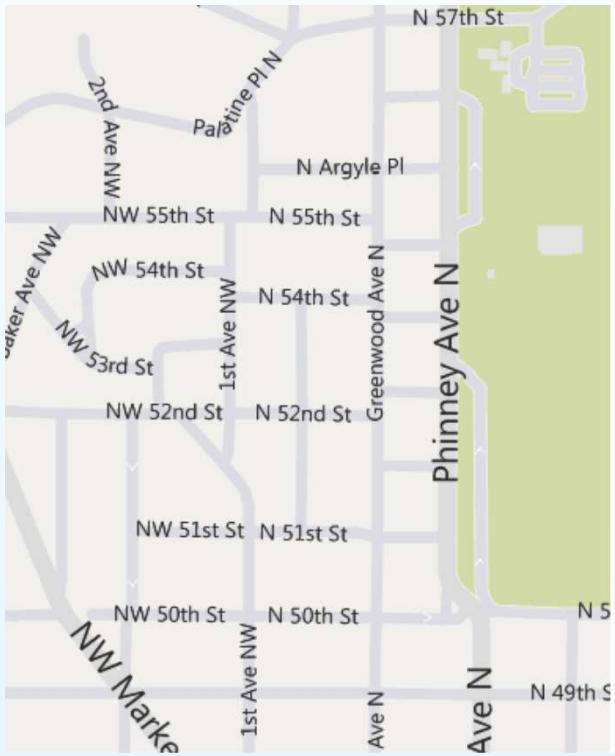

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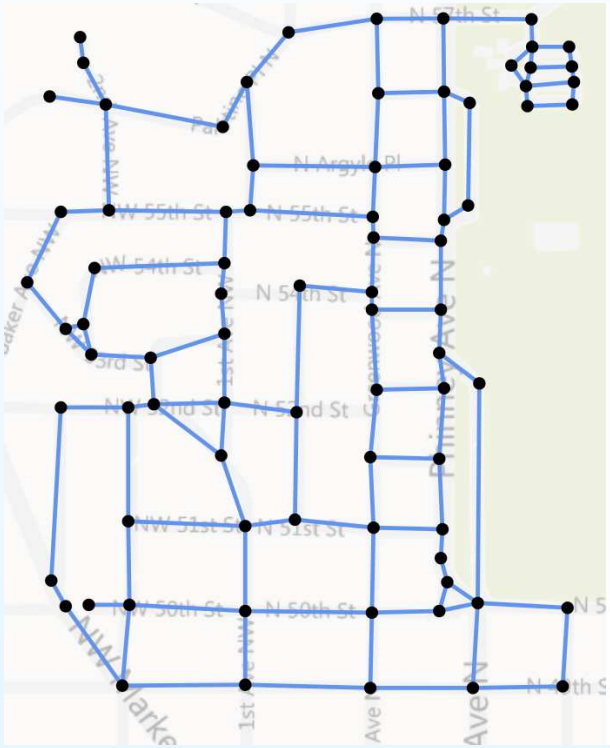

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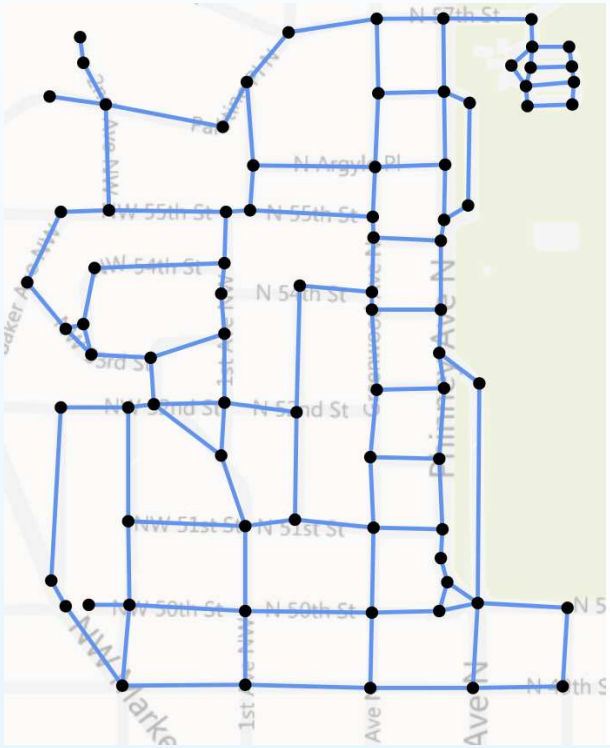

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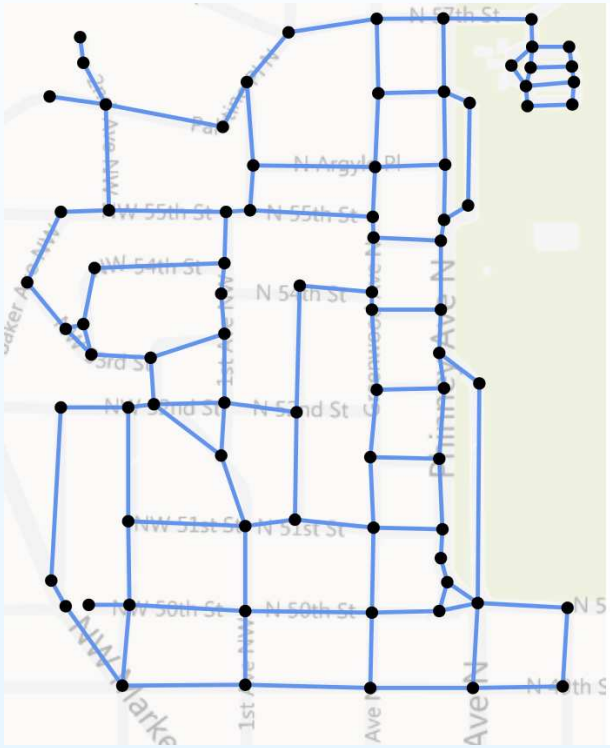

- Often, we want to model graphs that change over time
- Dynamic algorithms and data structures are tailor-made to handle changes in very large graphs

Graphs

- A graph G is a pair (V, E) where V is the set of *vertices* and E is the set of connections between these vertices called *edges*

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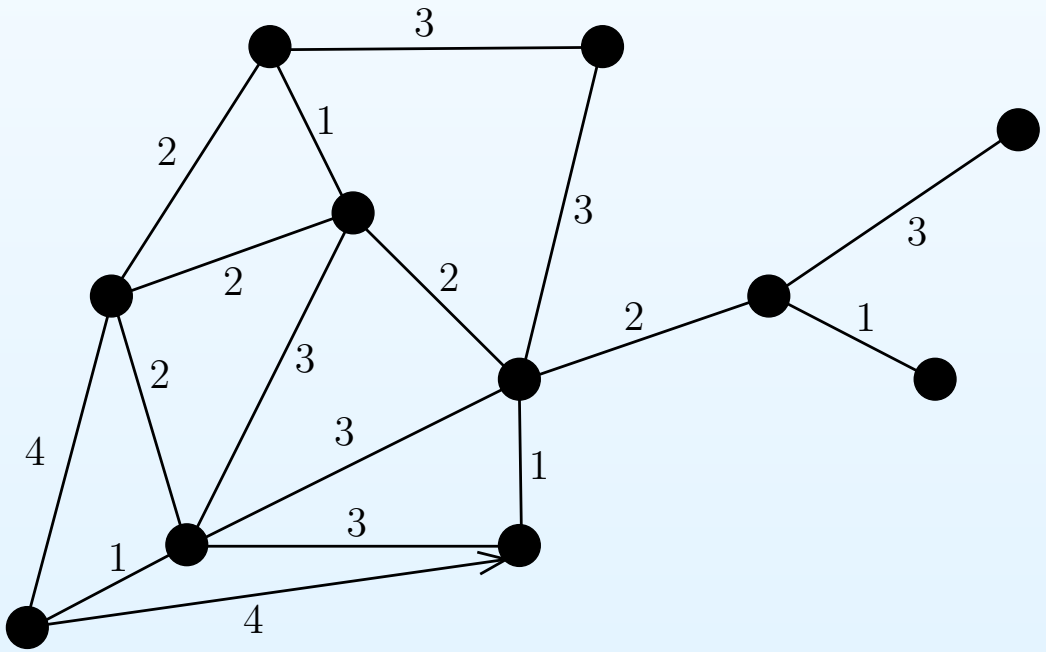
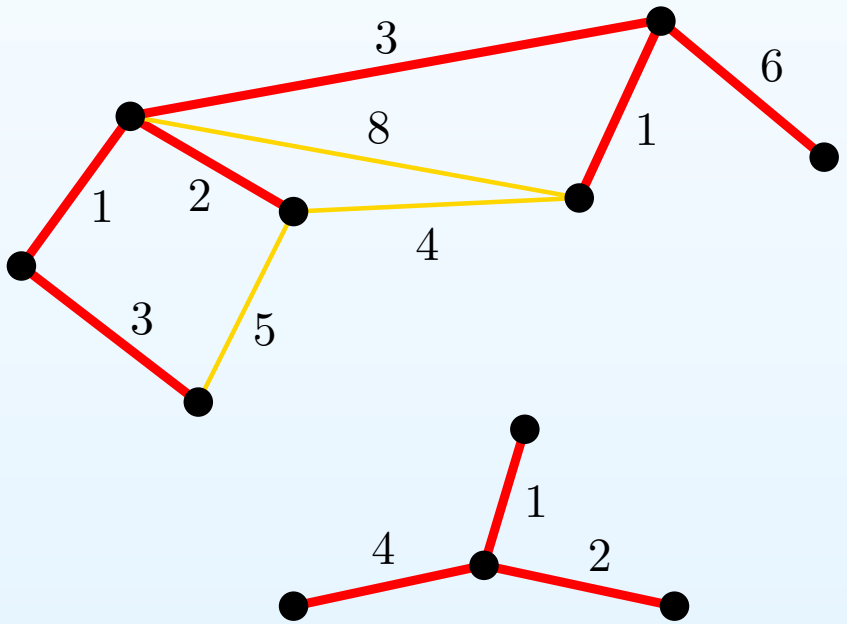
- Let $m = |E|$ and $n = |V|$ in the following
- Edges may have weights and may be directed or undirected:



A classical graph problem: minimum spanning forest

- We have near-linear time algorithms (Prim, Kruskal,...) to compute a minimum spanning forest of a given graph:



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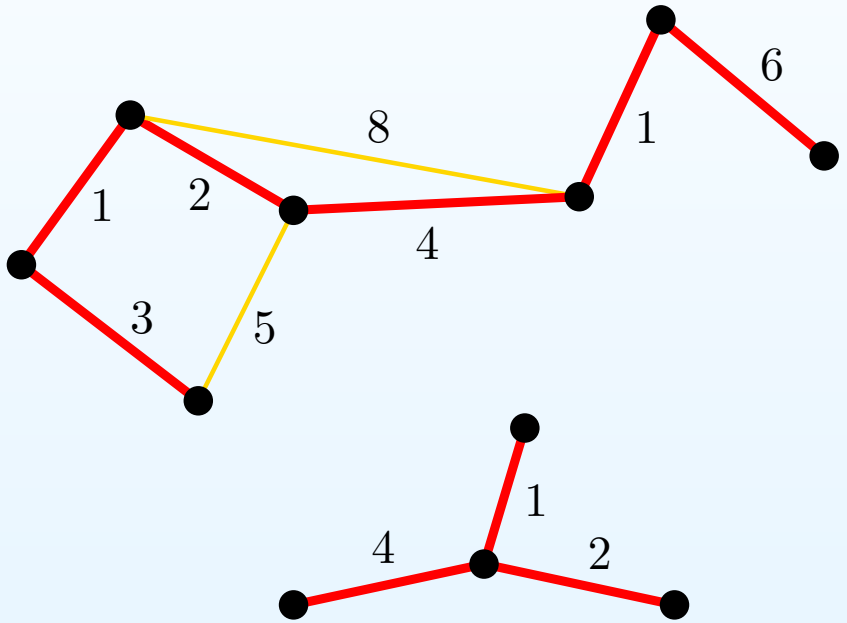
- Maintaining a minimum spanning forest in a dynamic graph:



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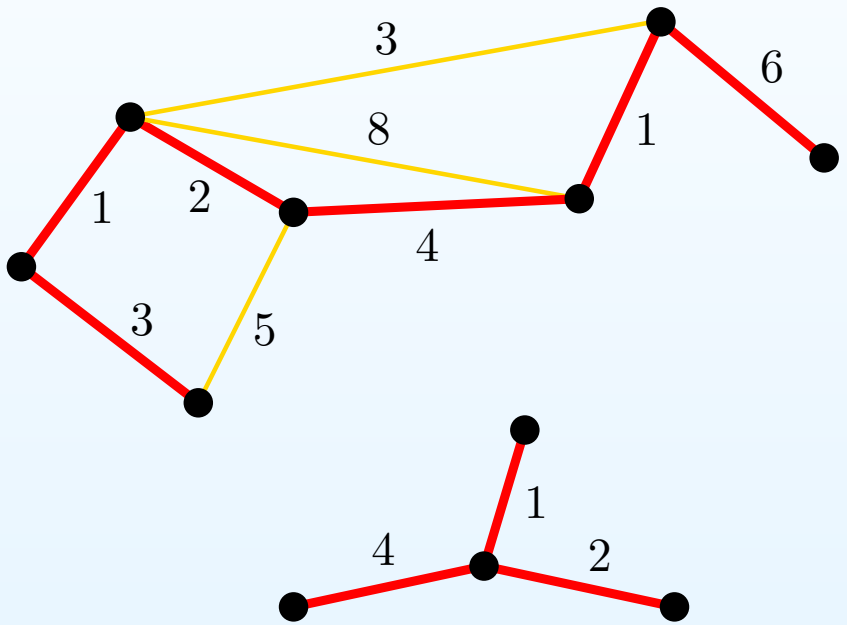

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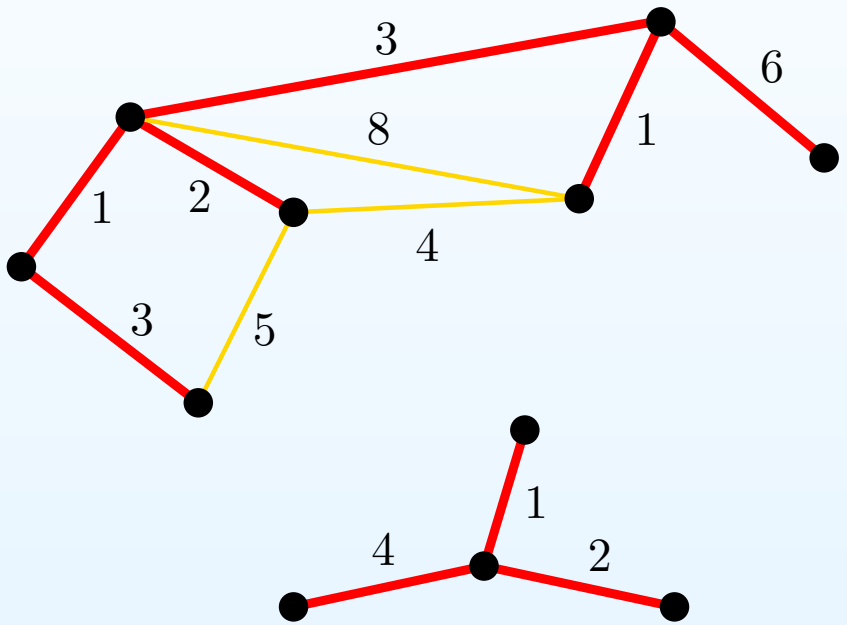

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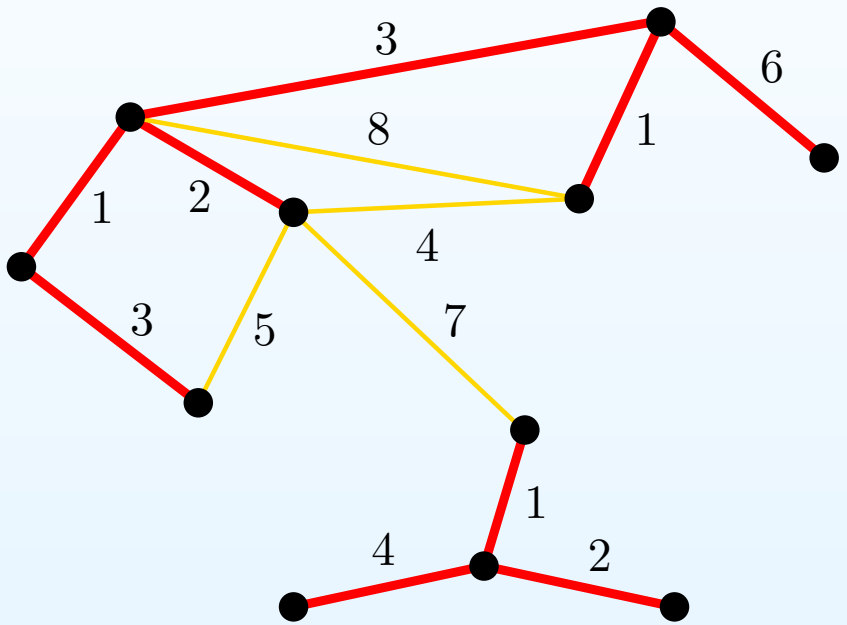

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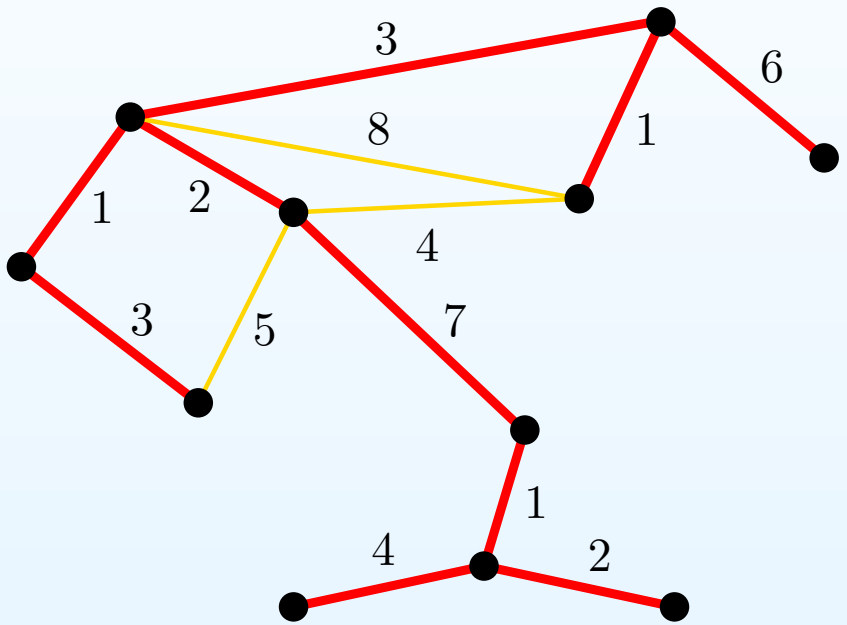

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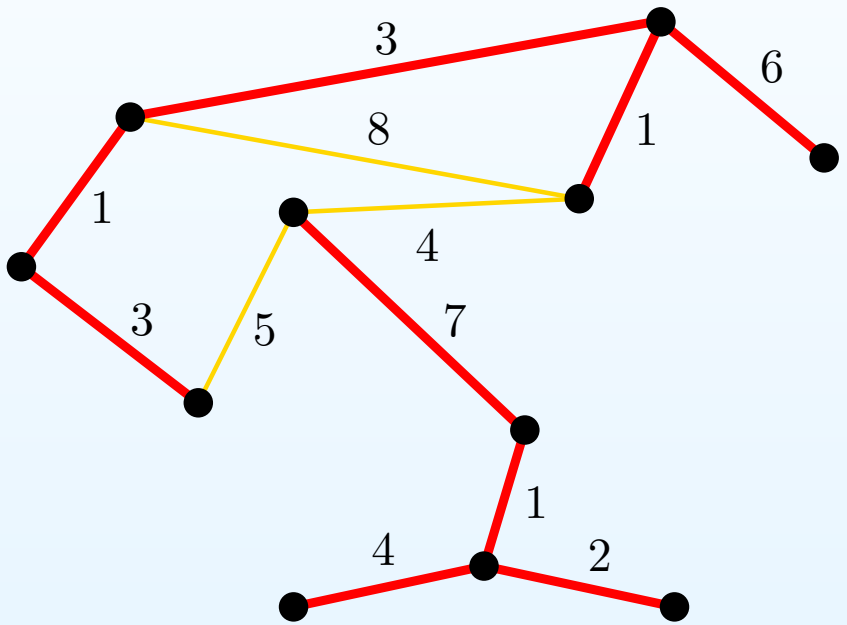

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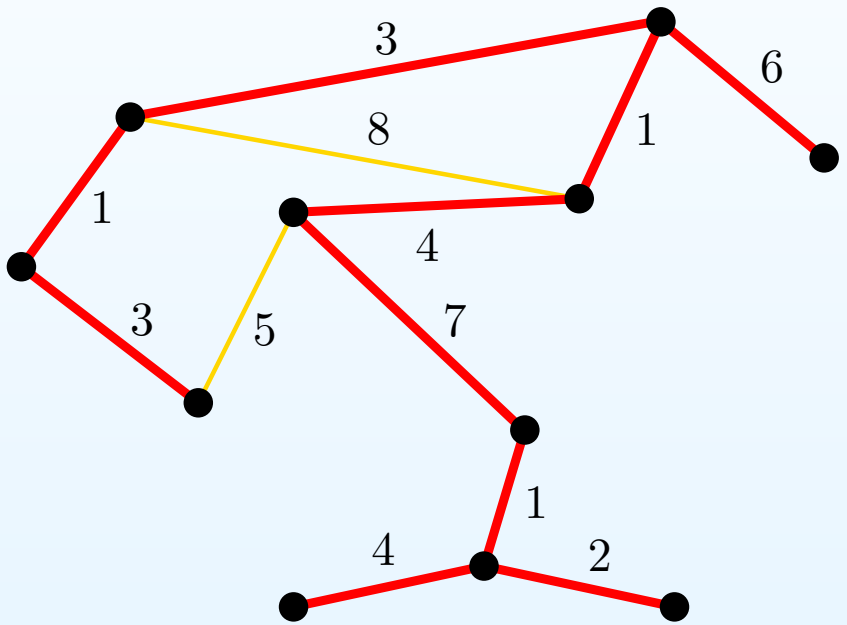

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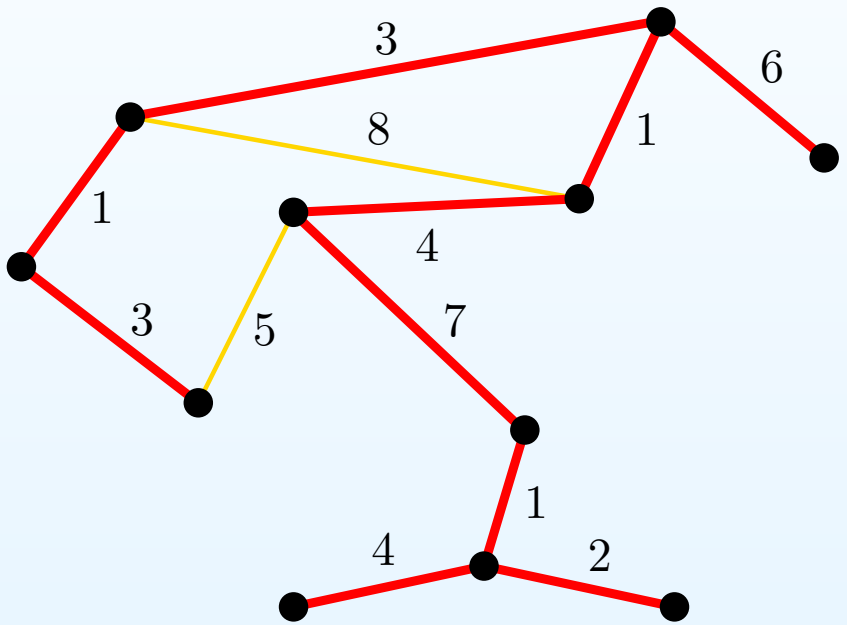

- Can we handle each update faster than rebuilding from scratch?

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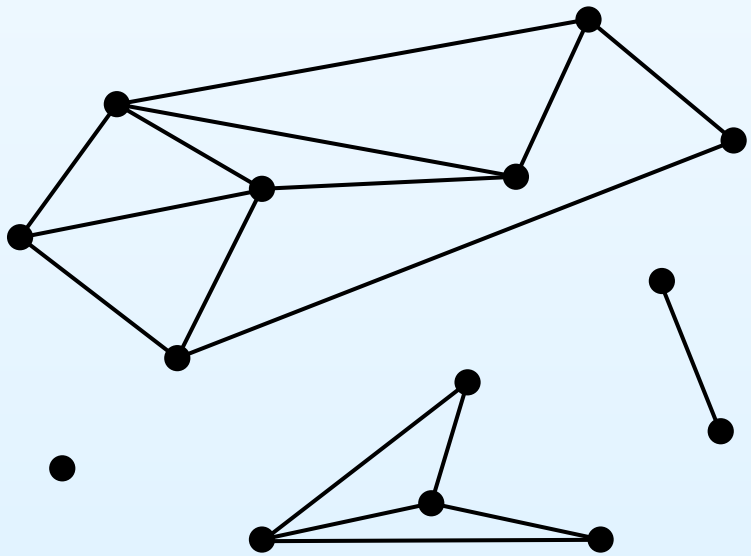

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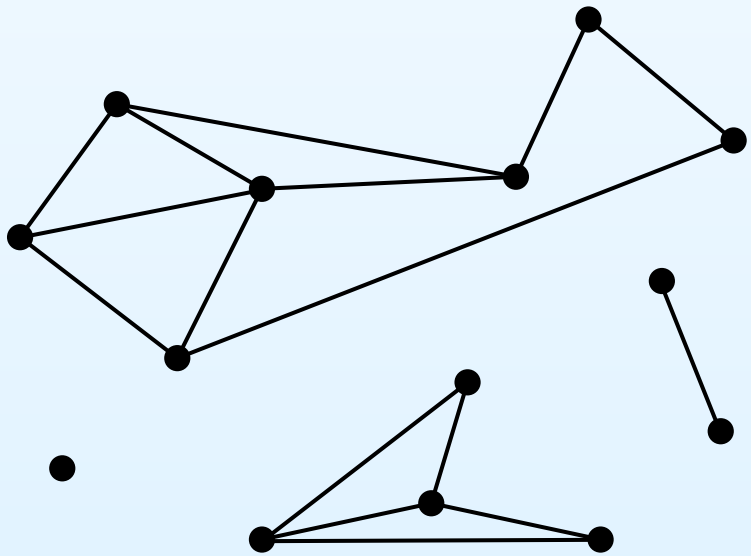

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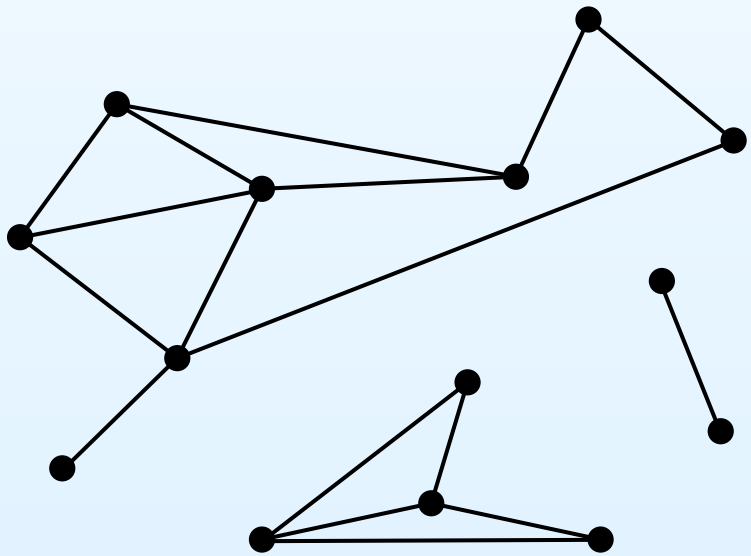

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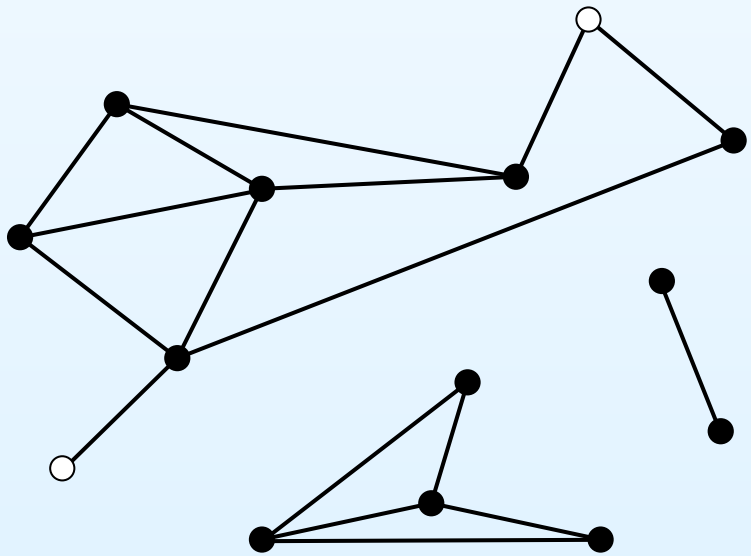

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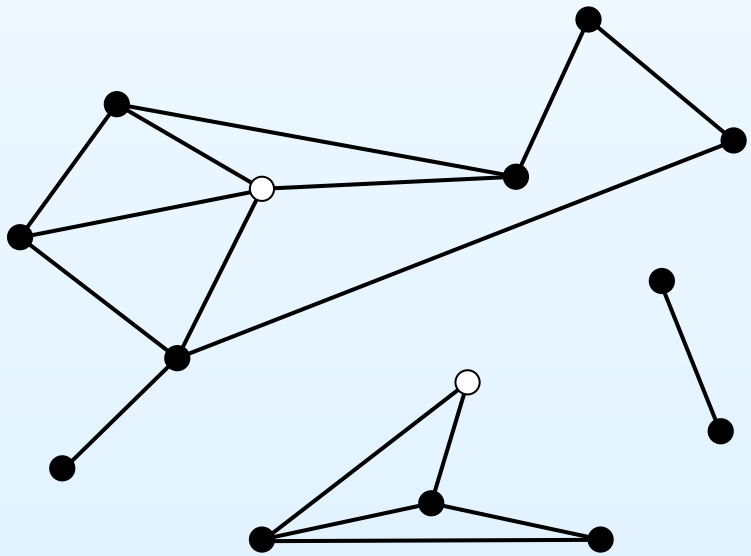
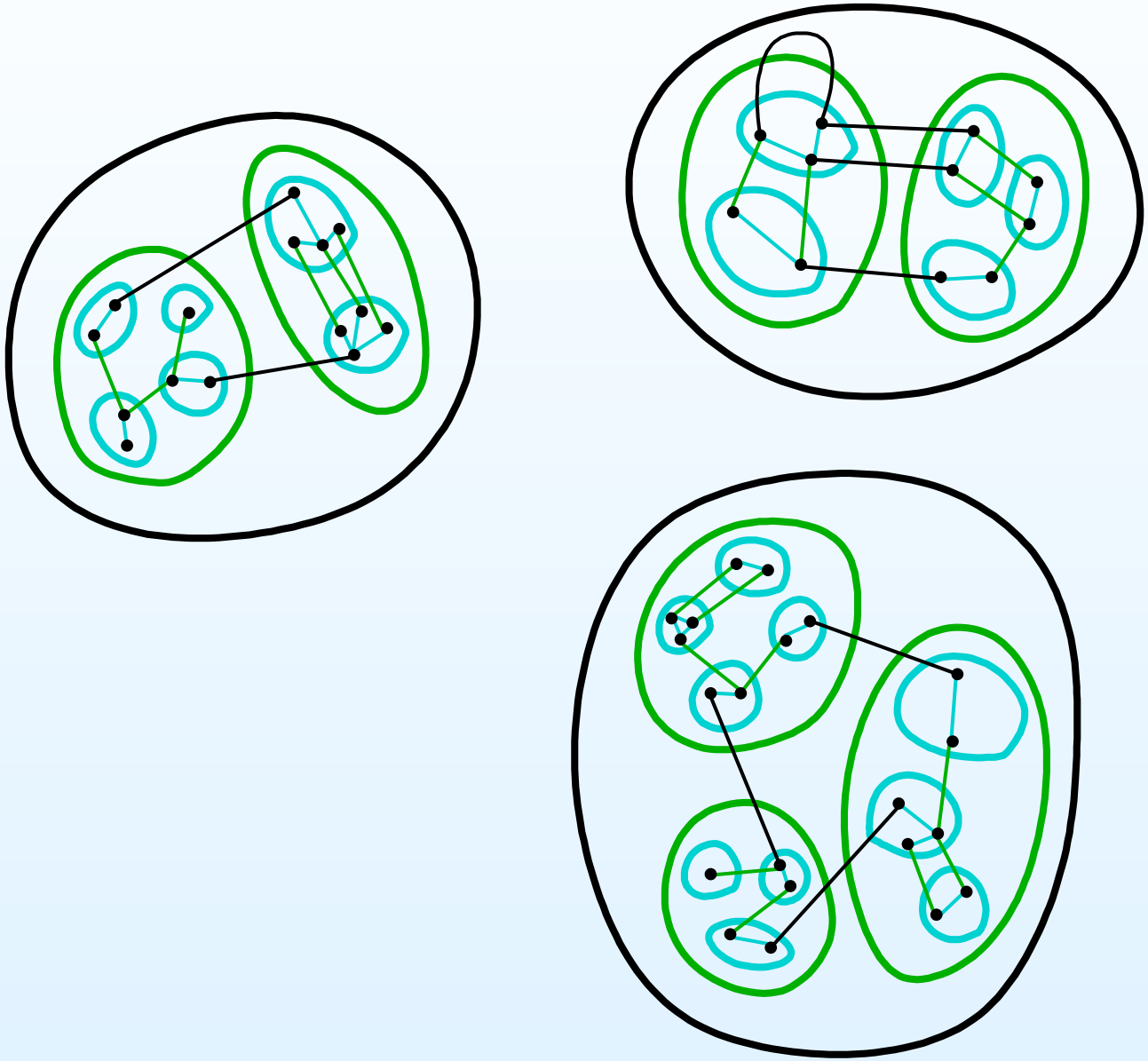
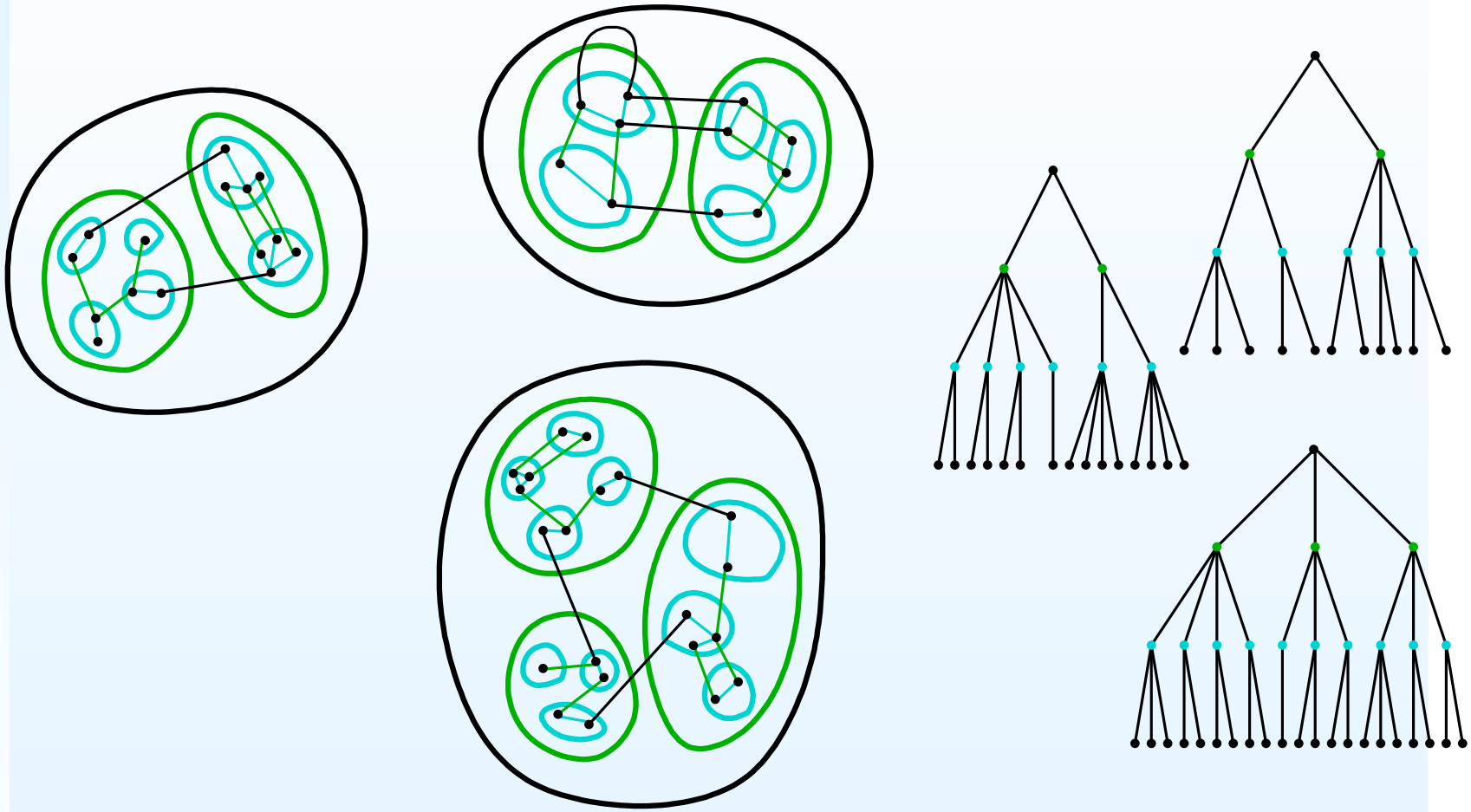


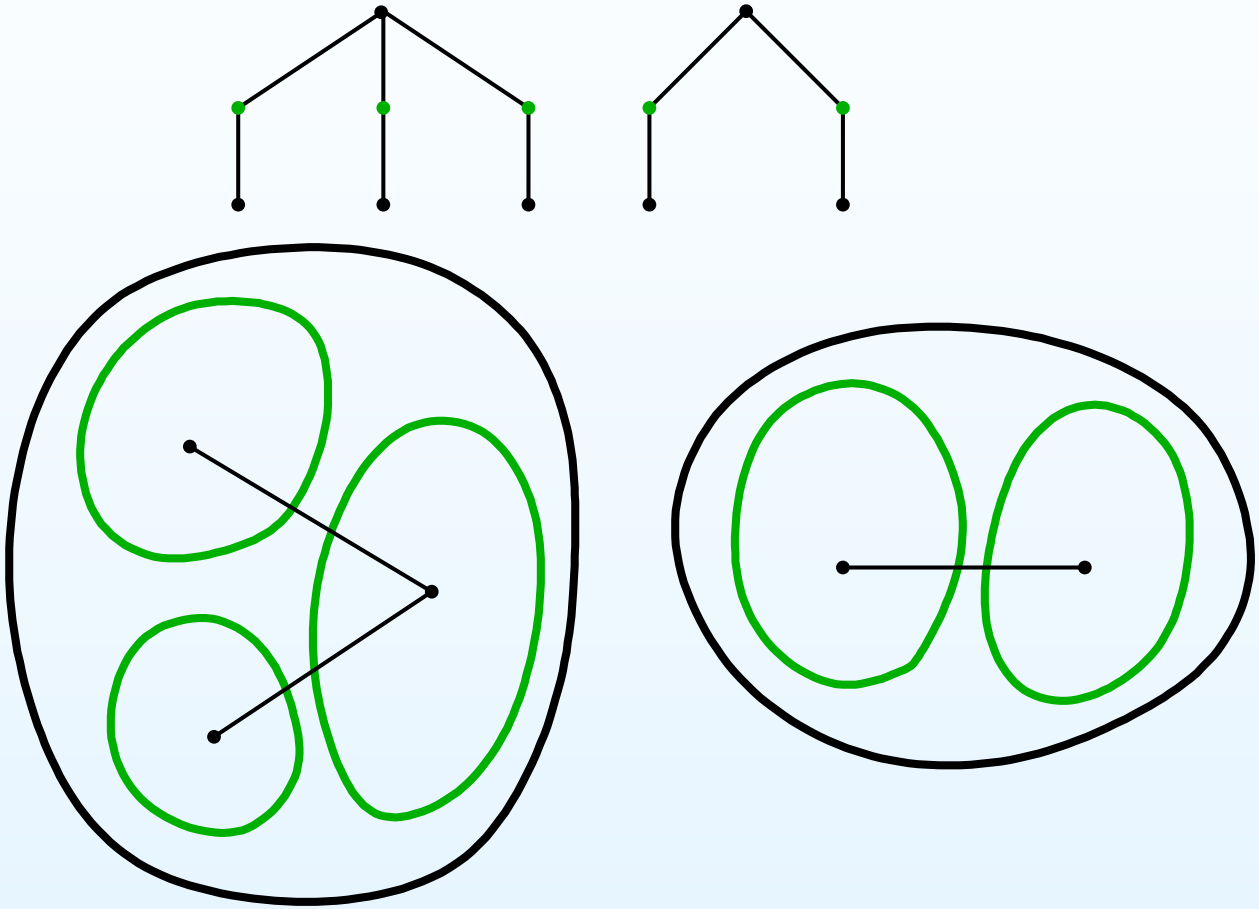
Clusters



Cluster forest



Handling an update $\texttt{insert}(u, v)$



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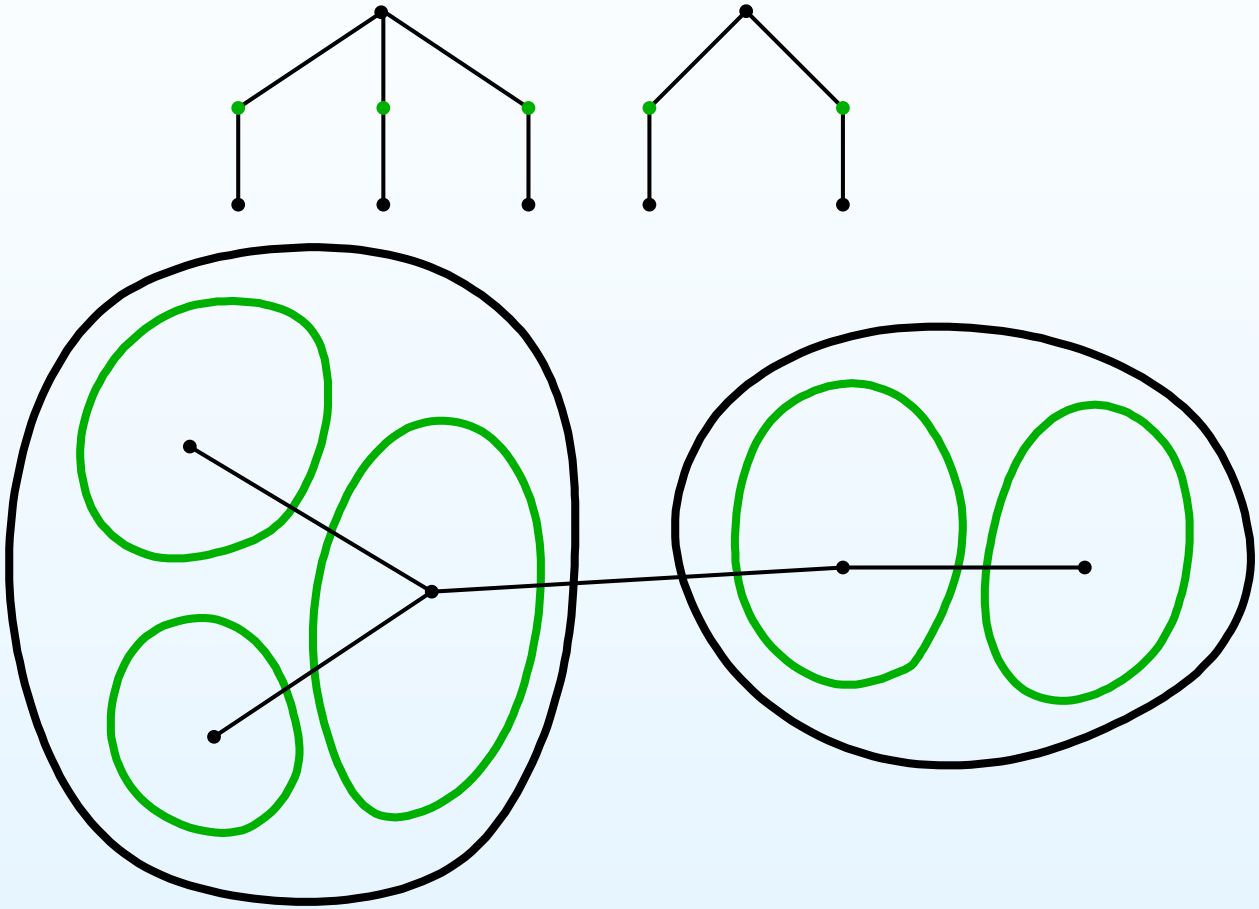

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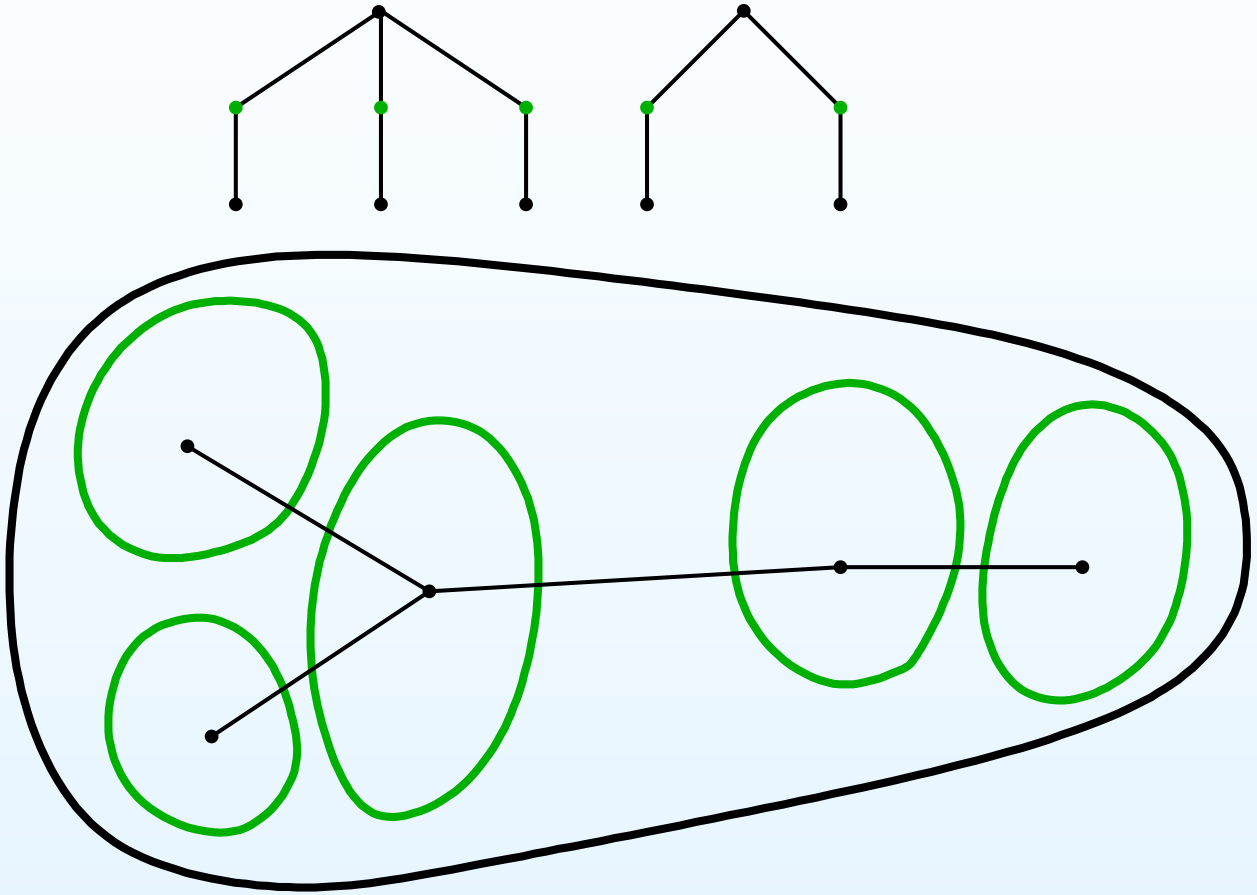

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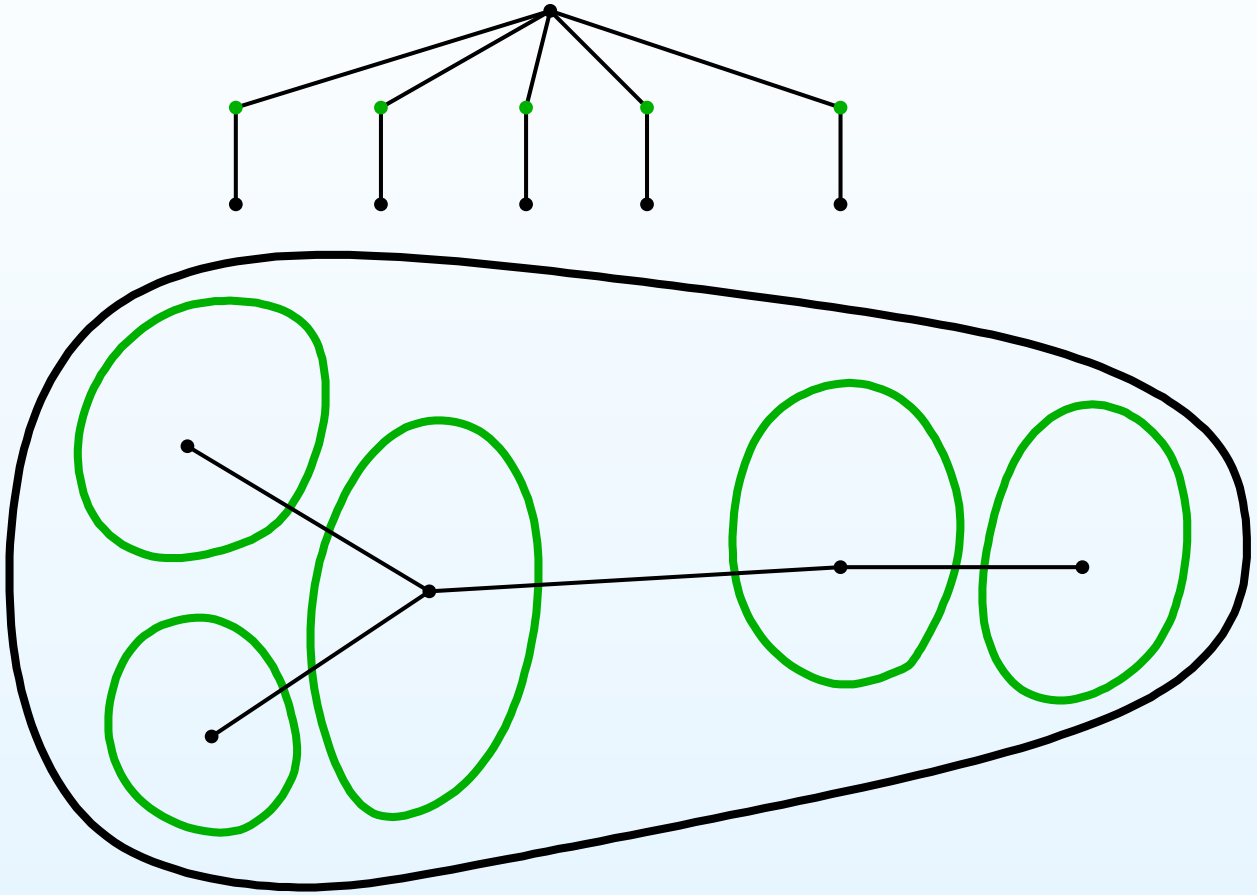
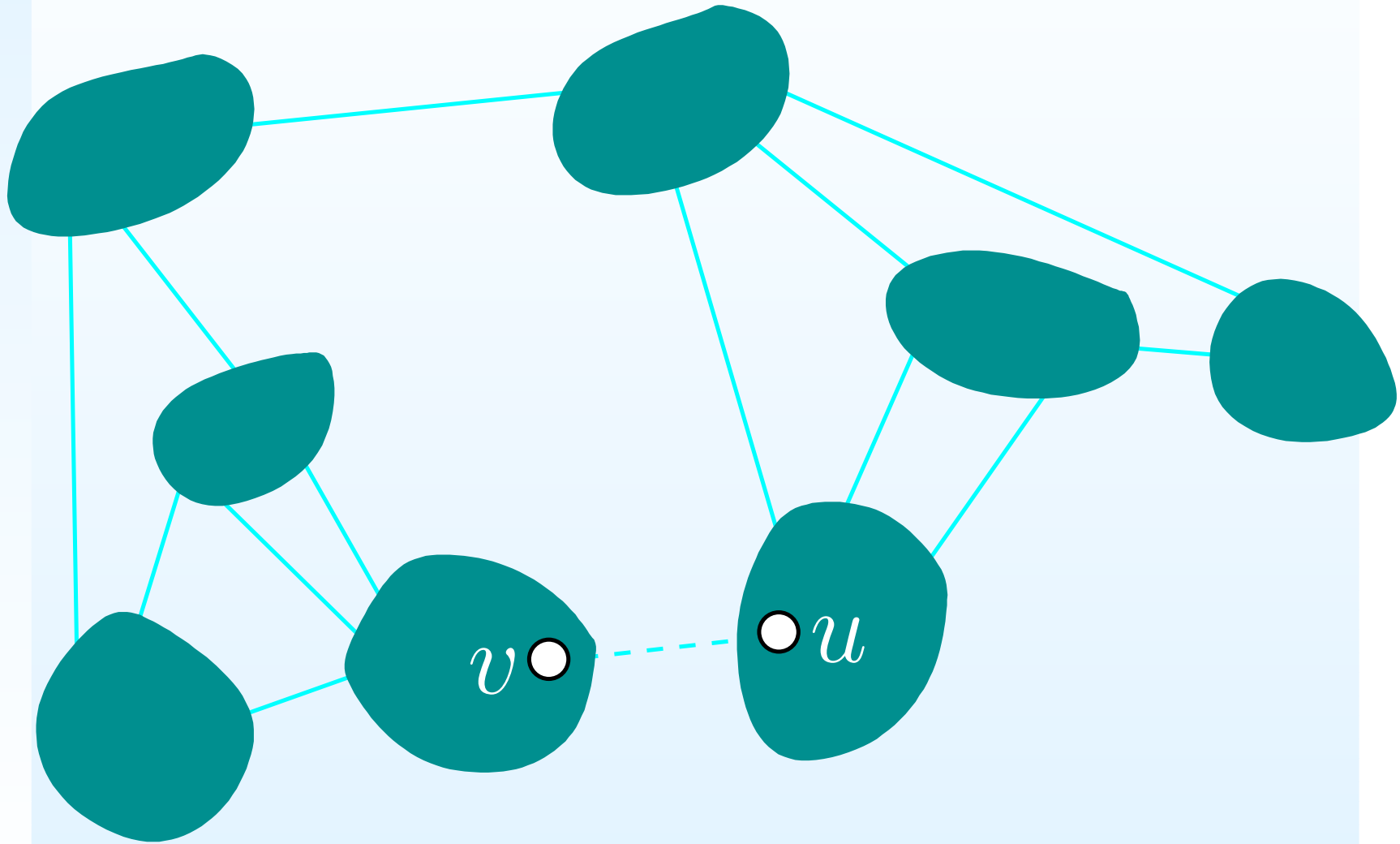


A vertex of M_i is explored by both search procedures



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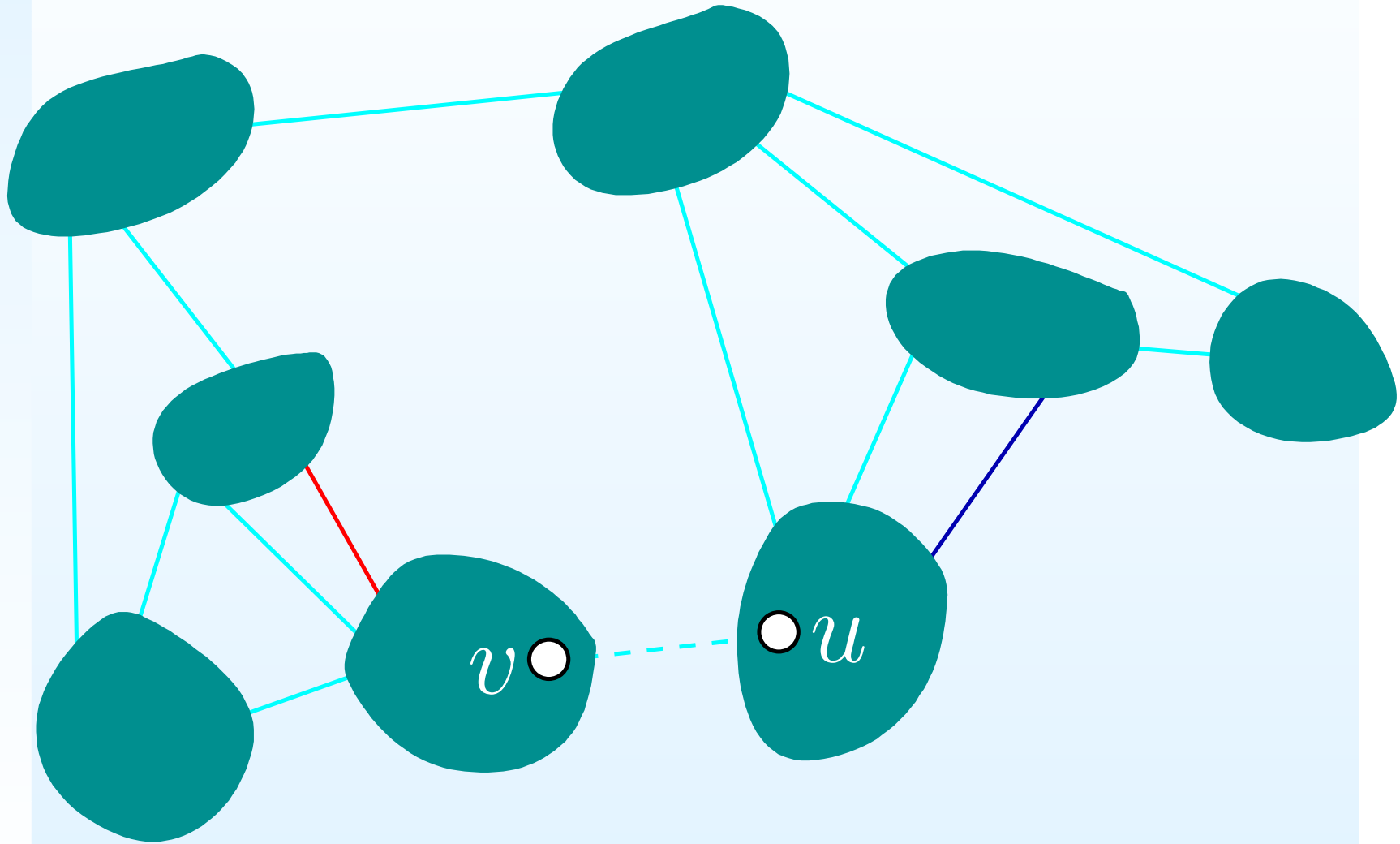

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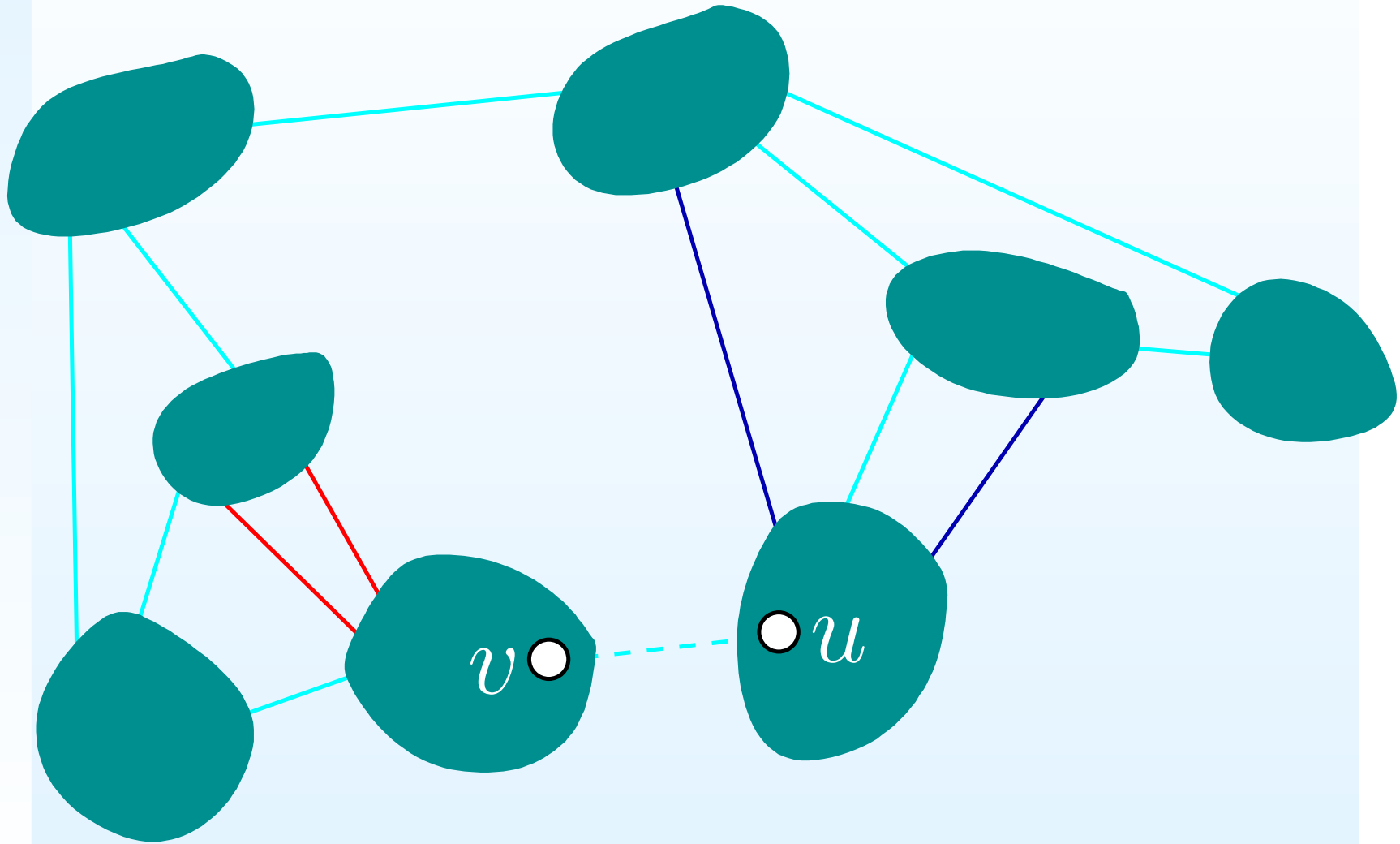

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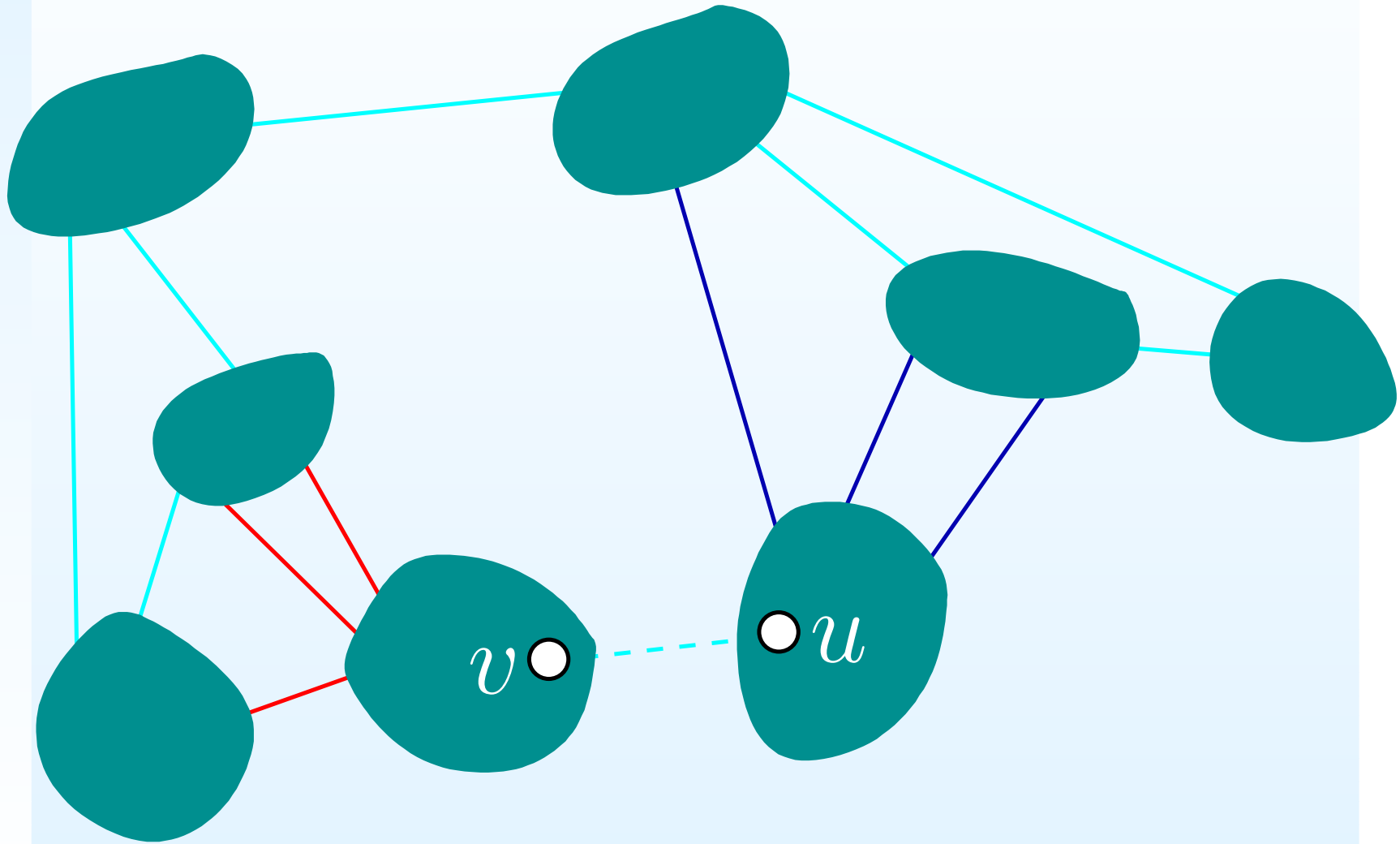

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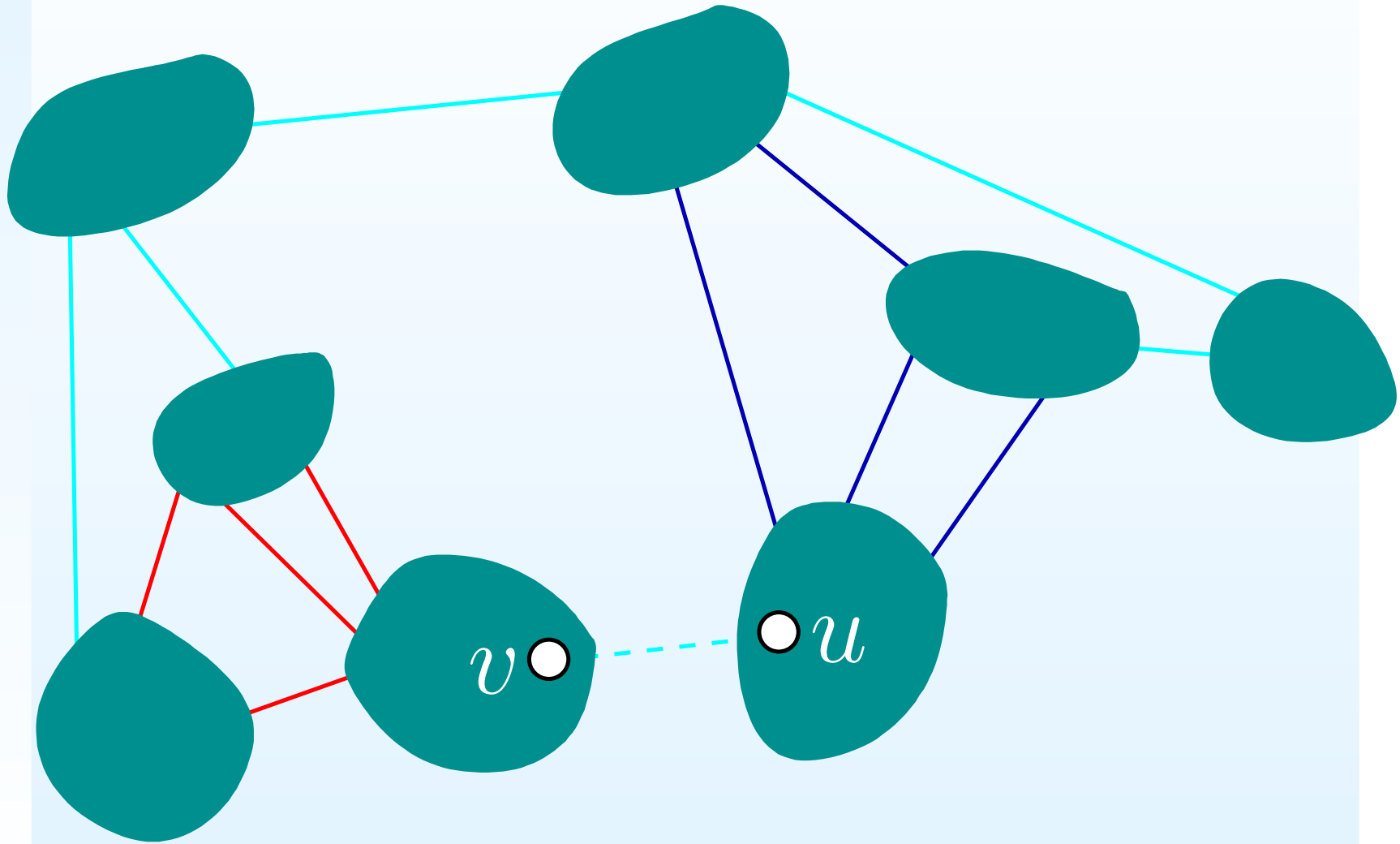

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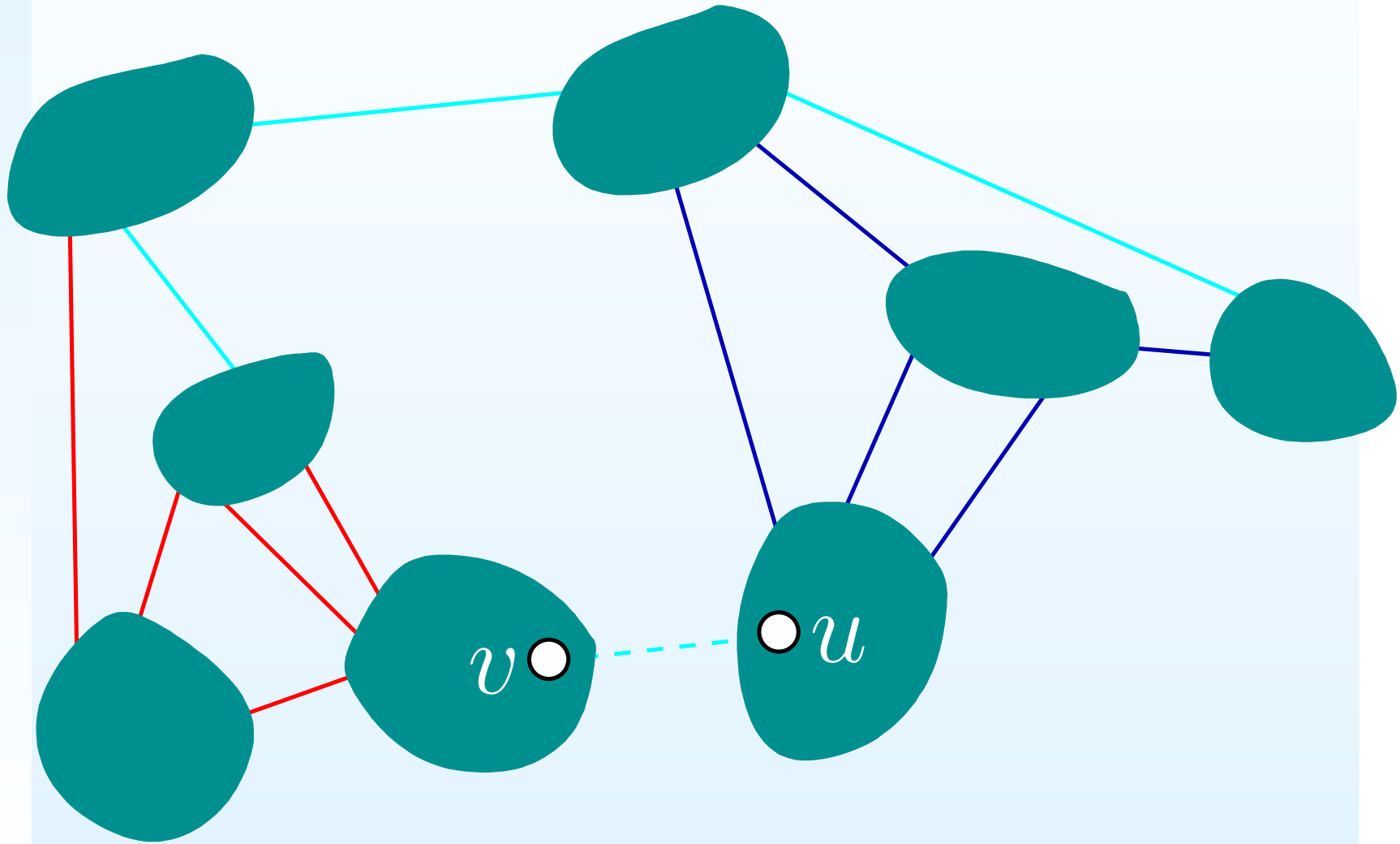

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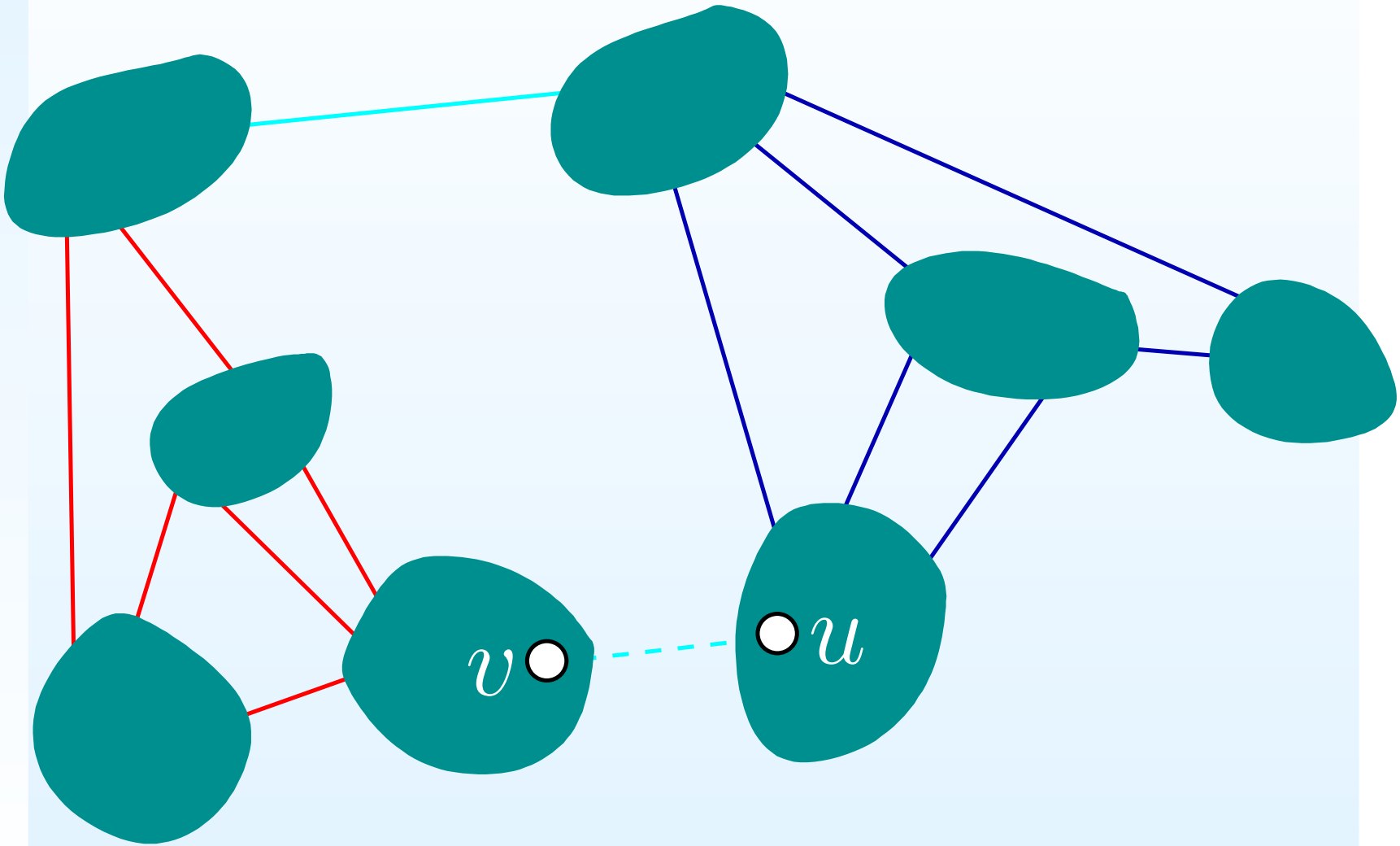

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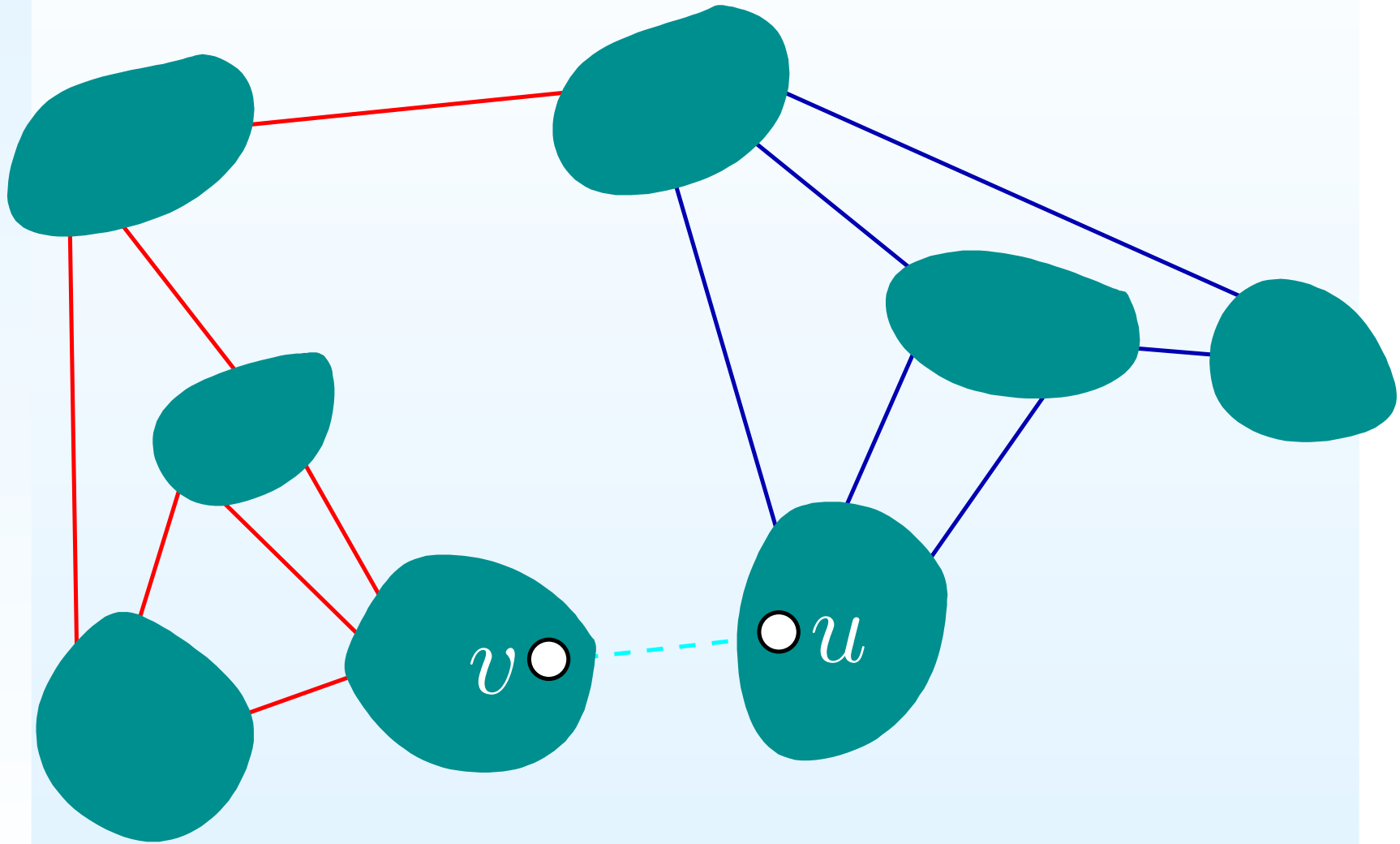

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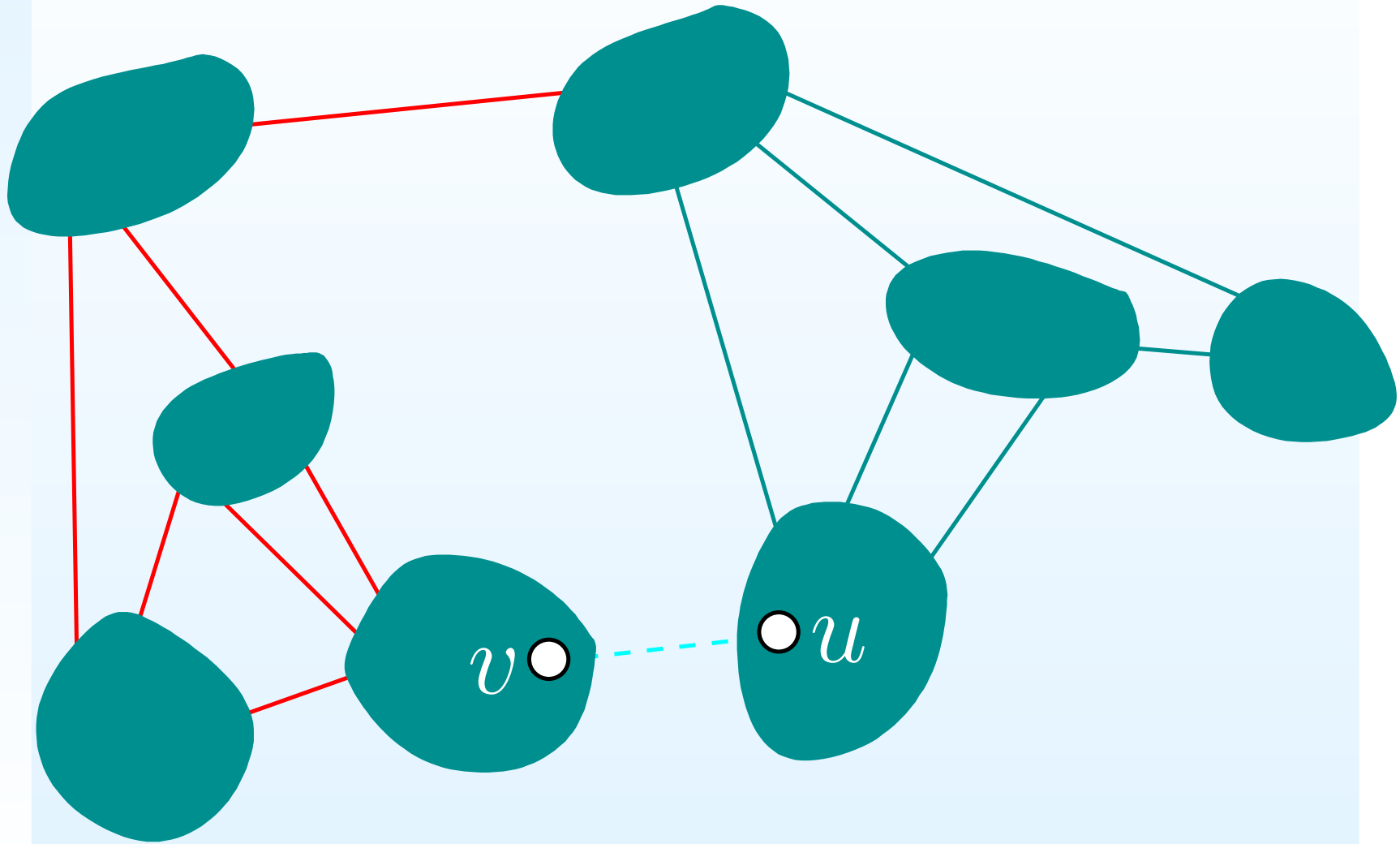

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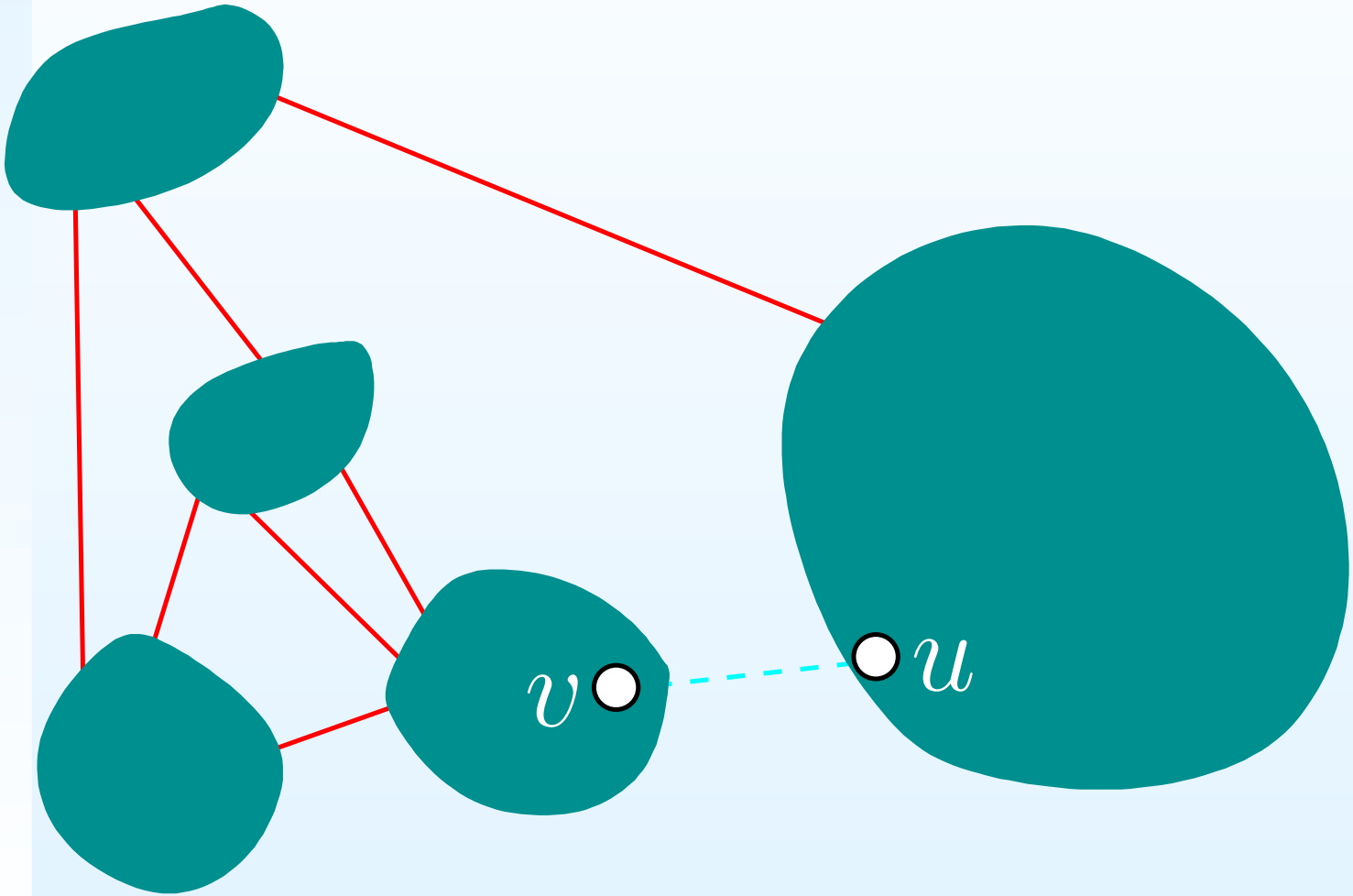

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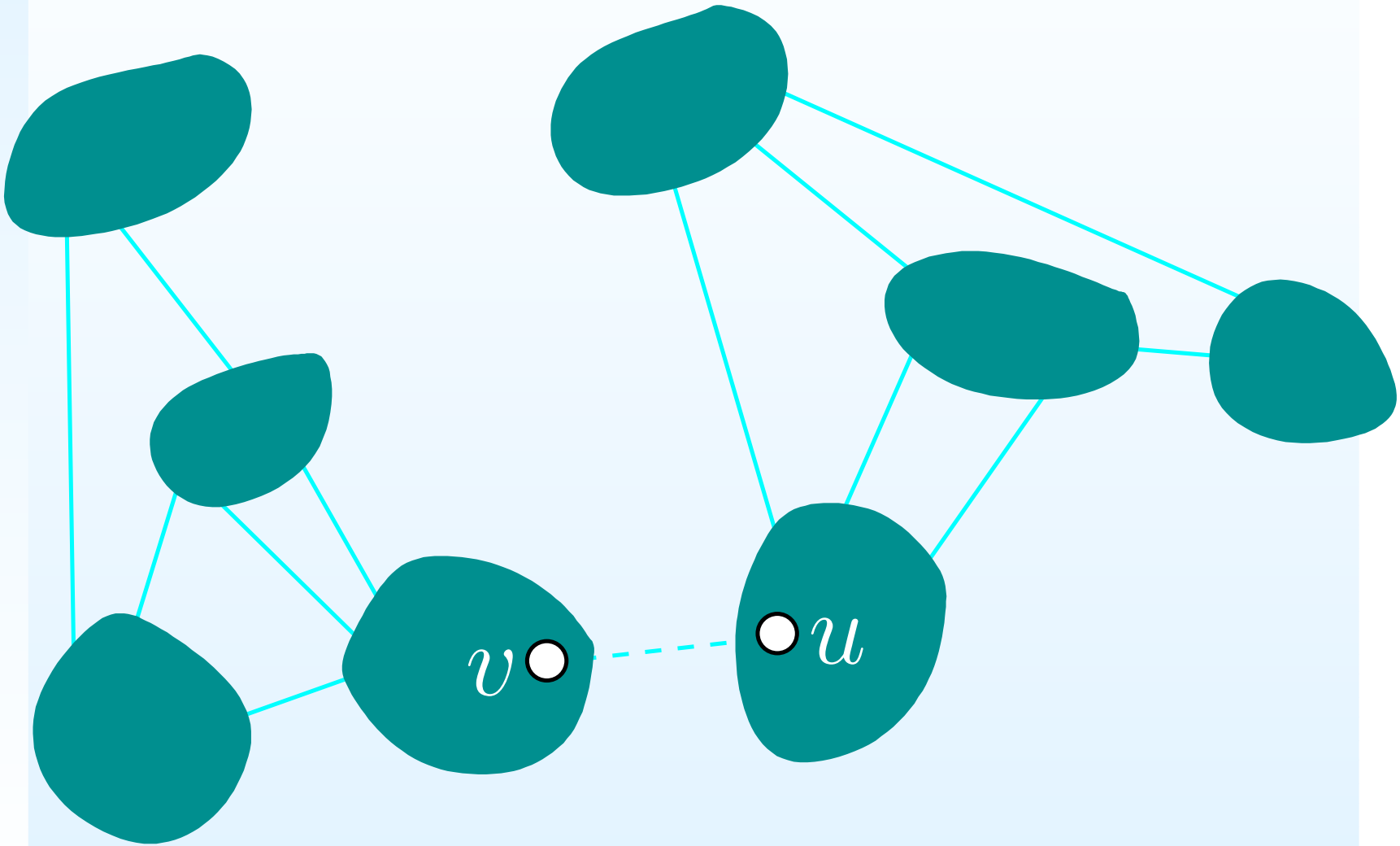

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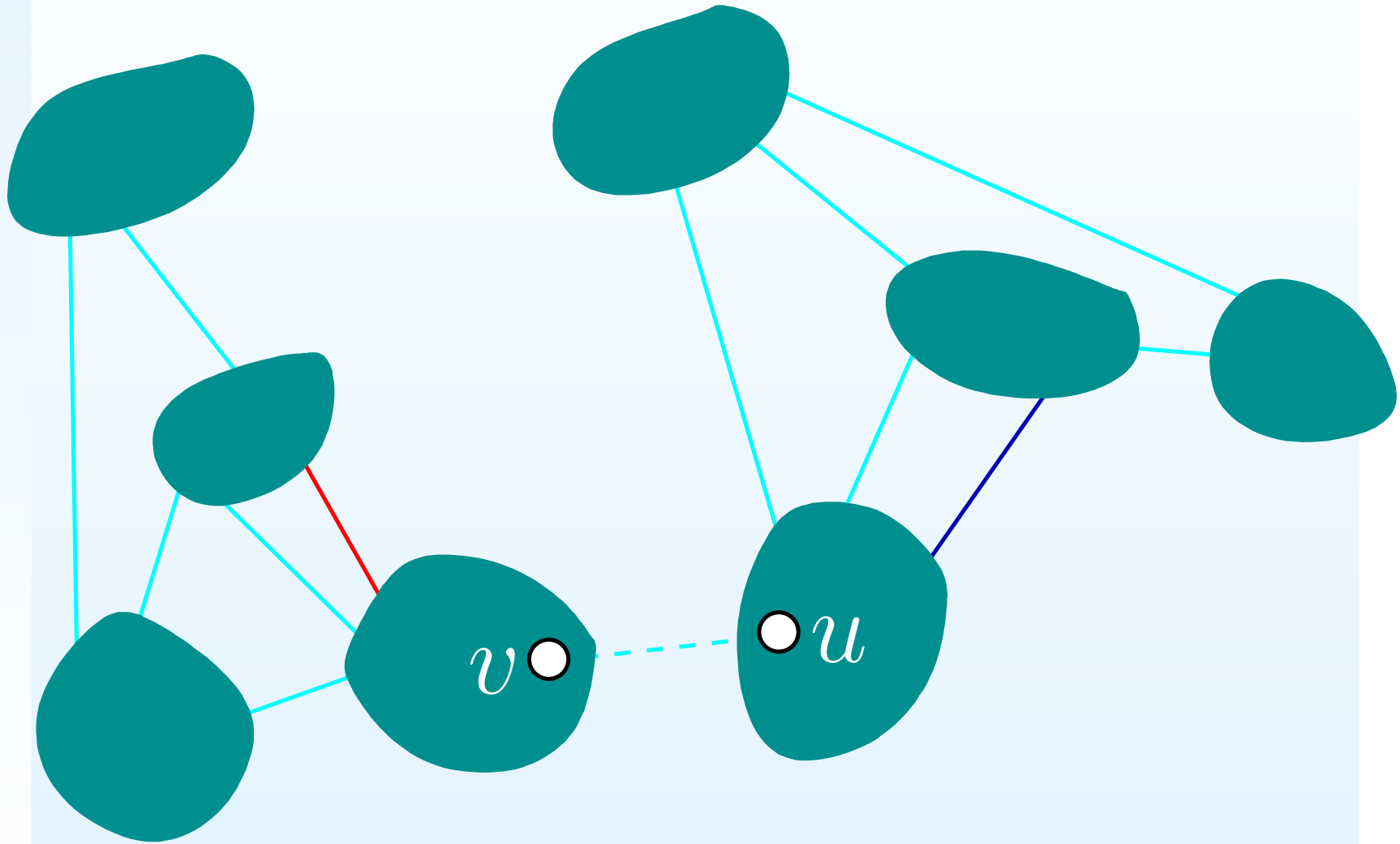

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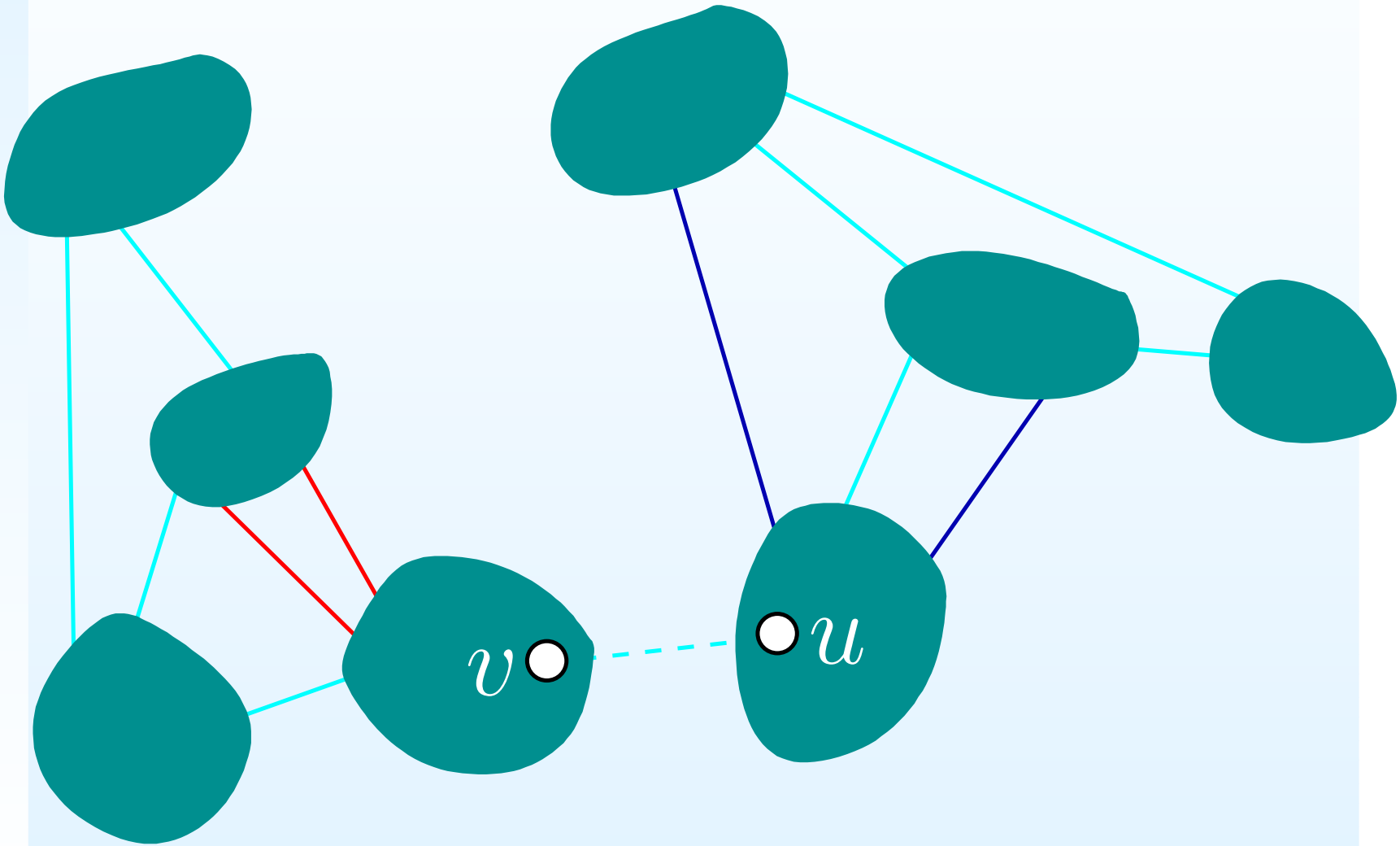

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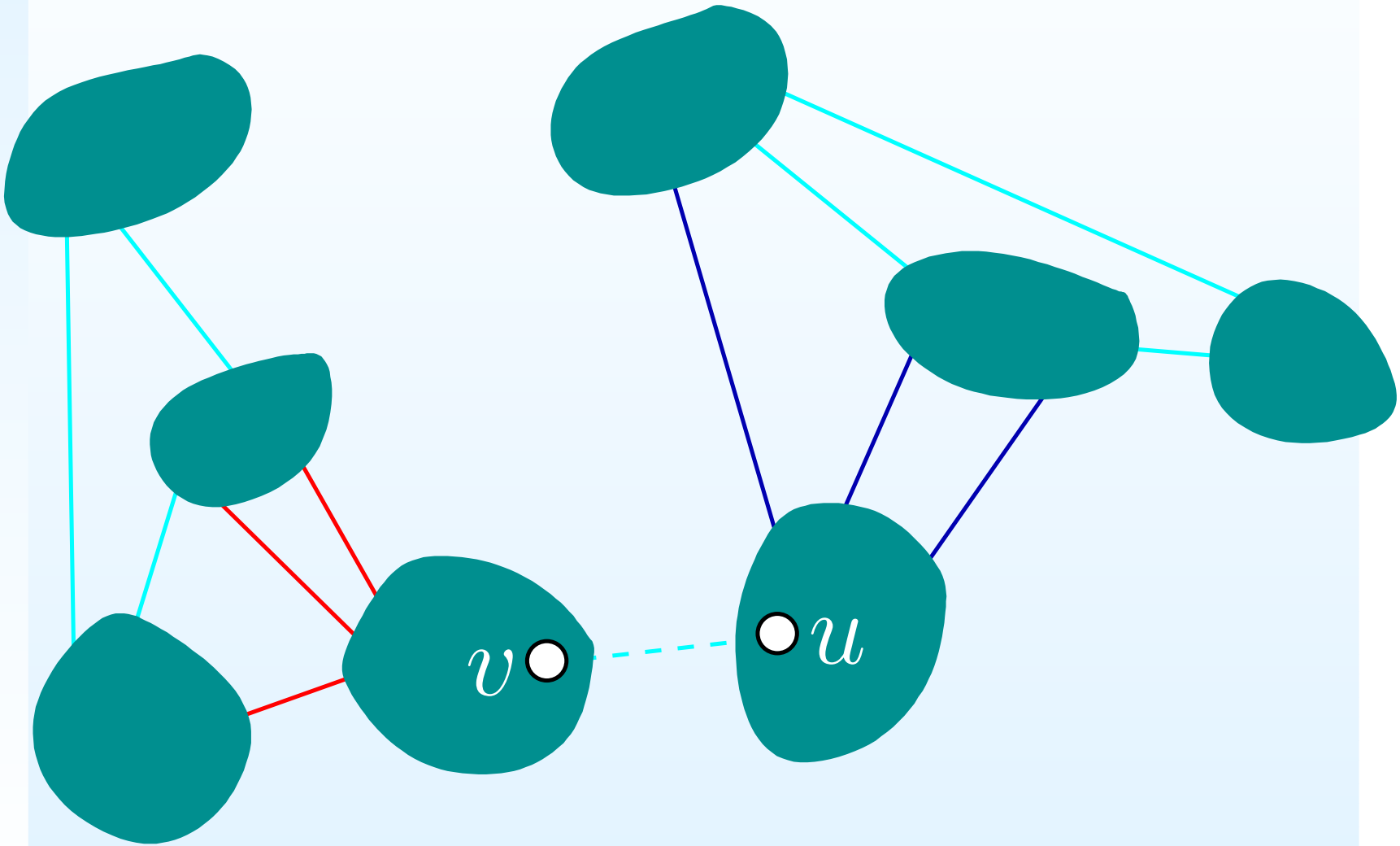

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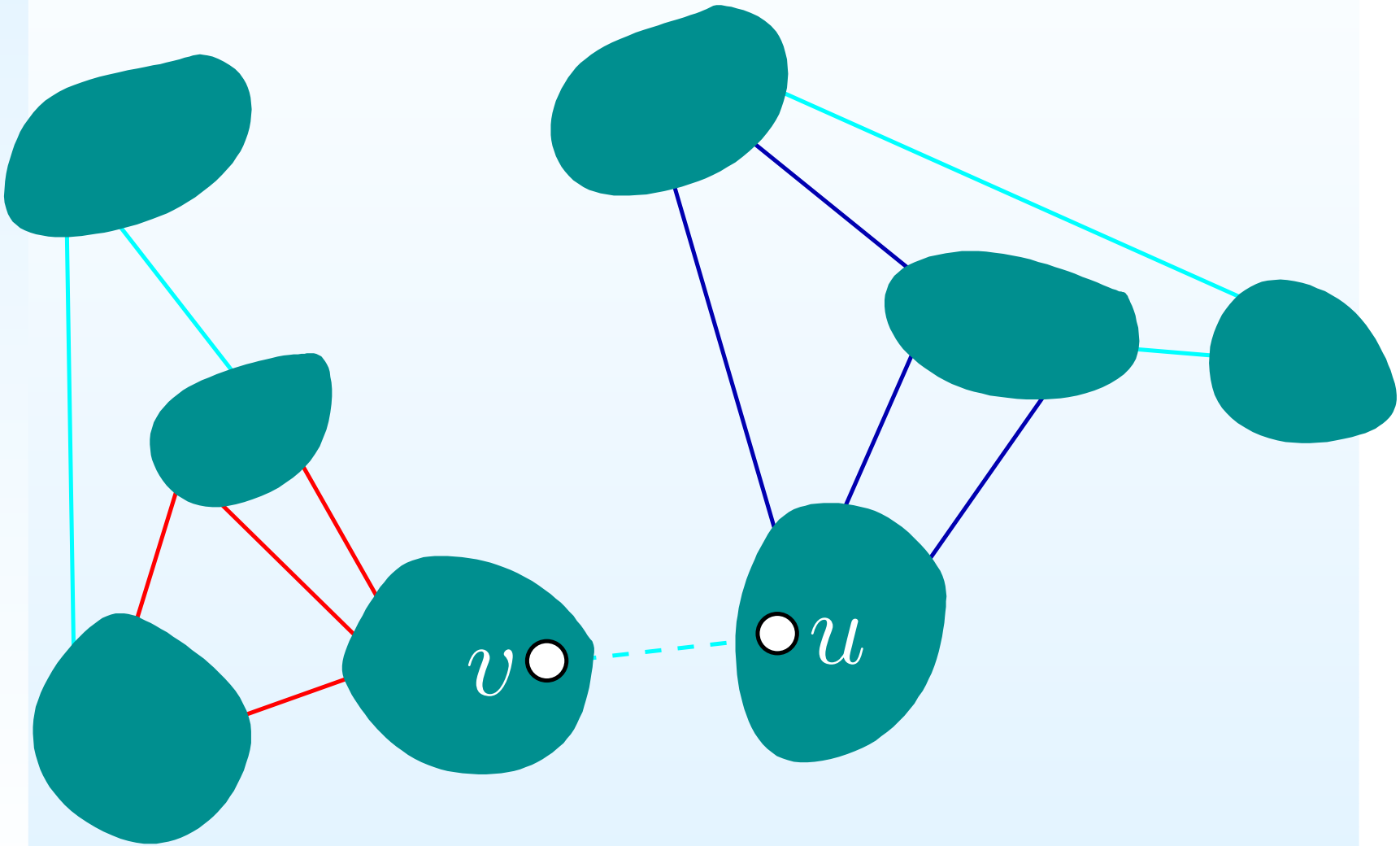

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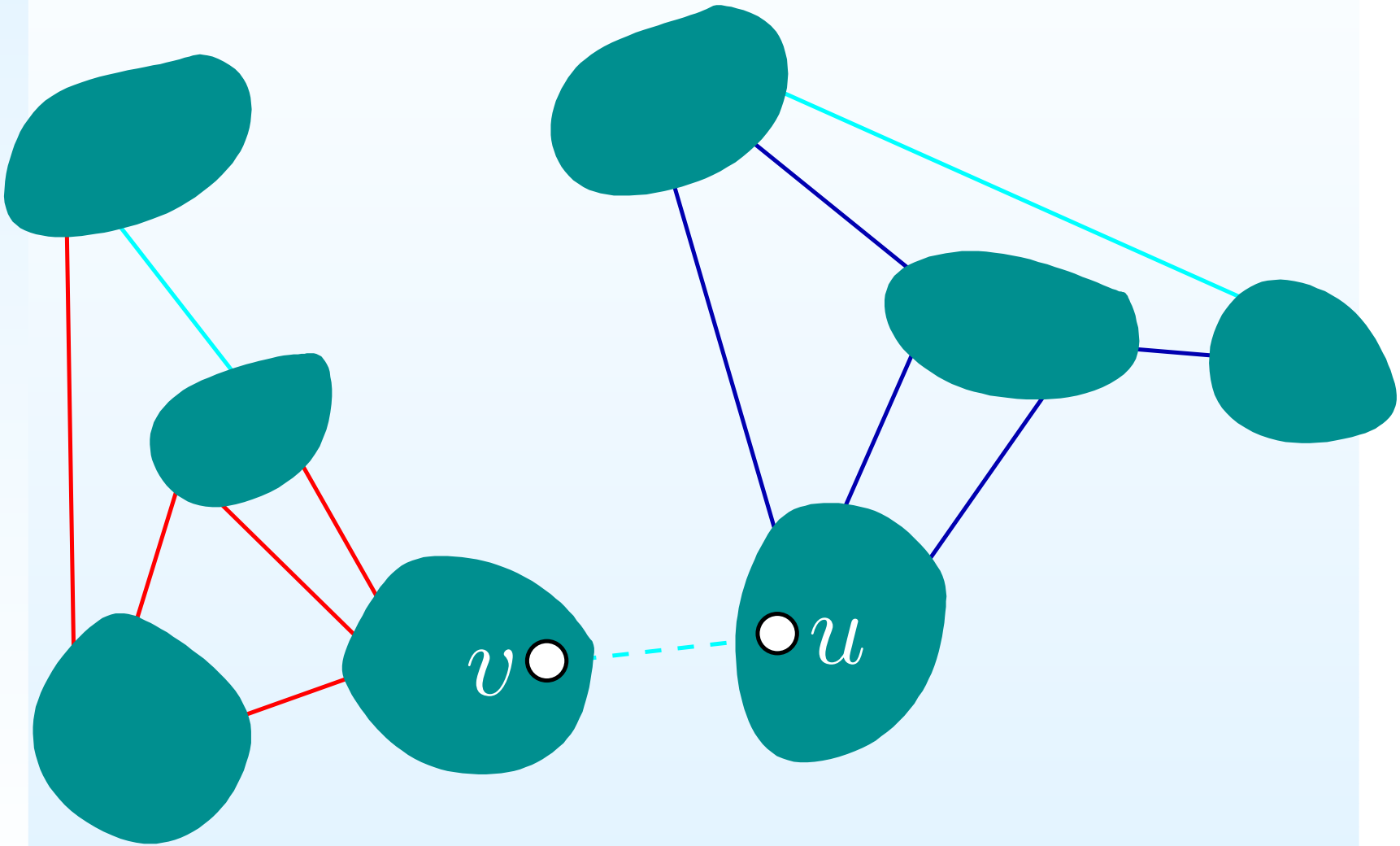

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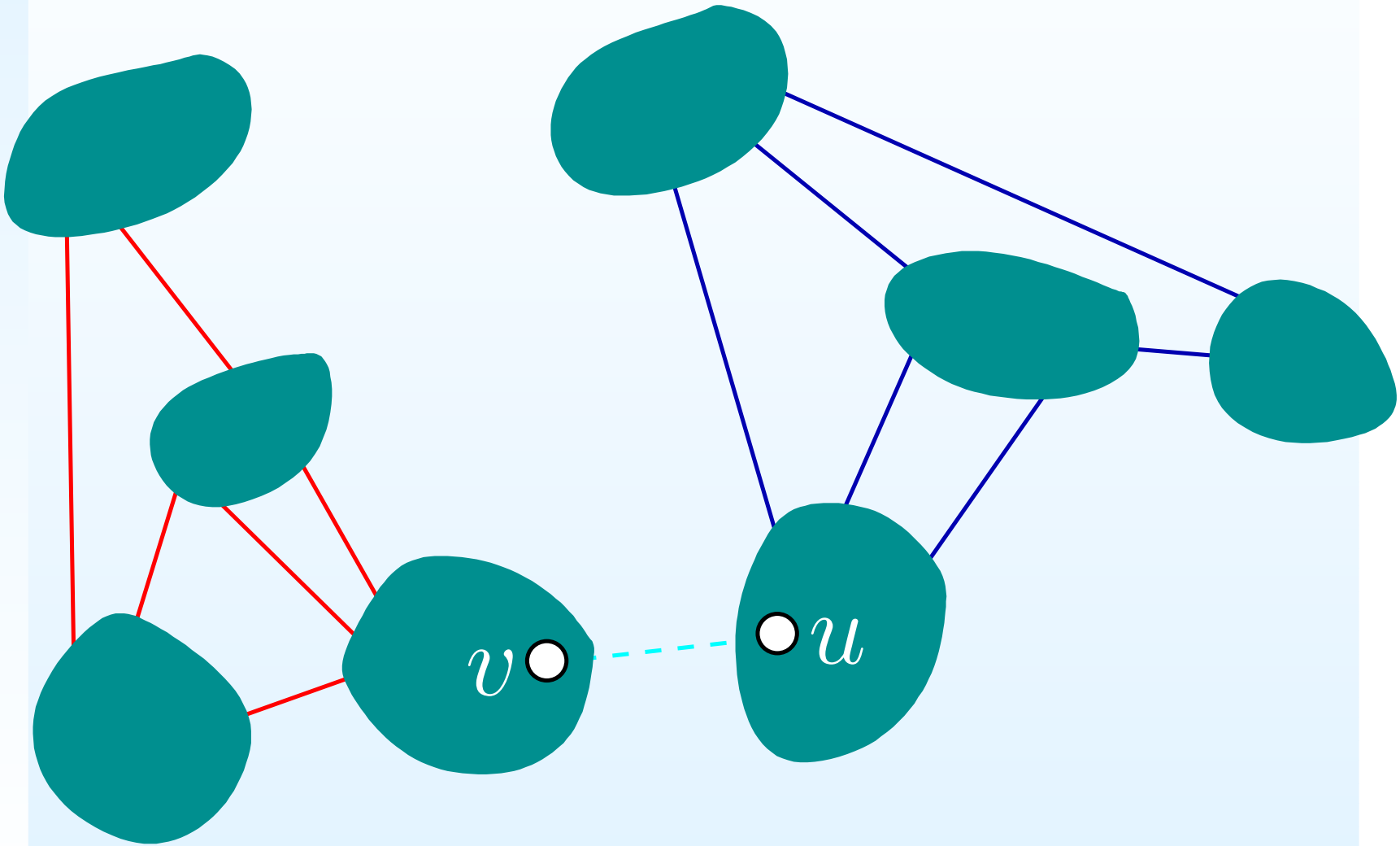

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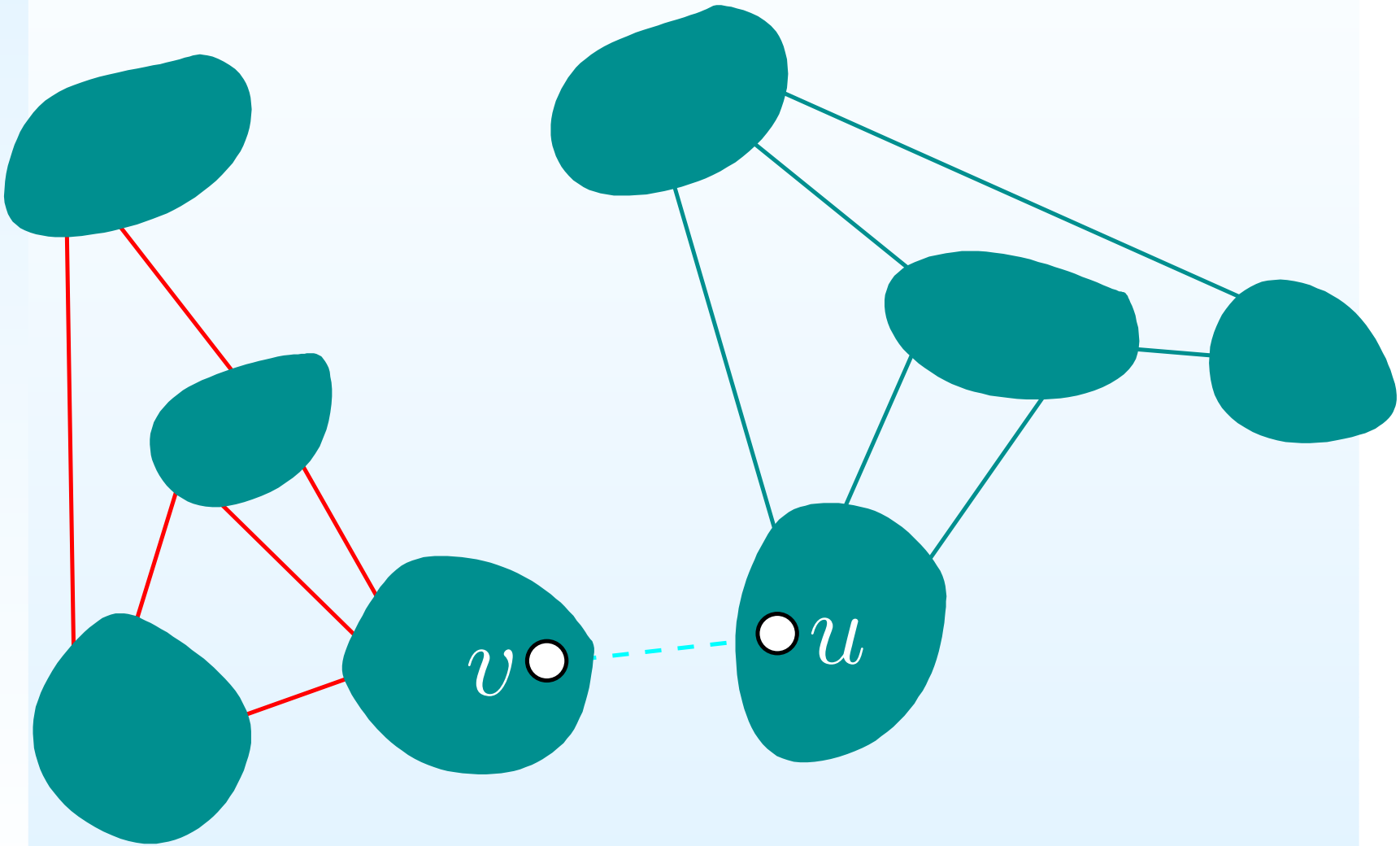

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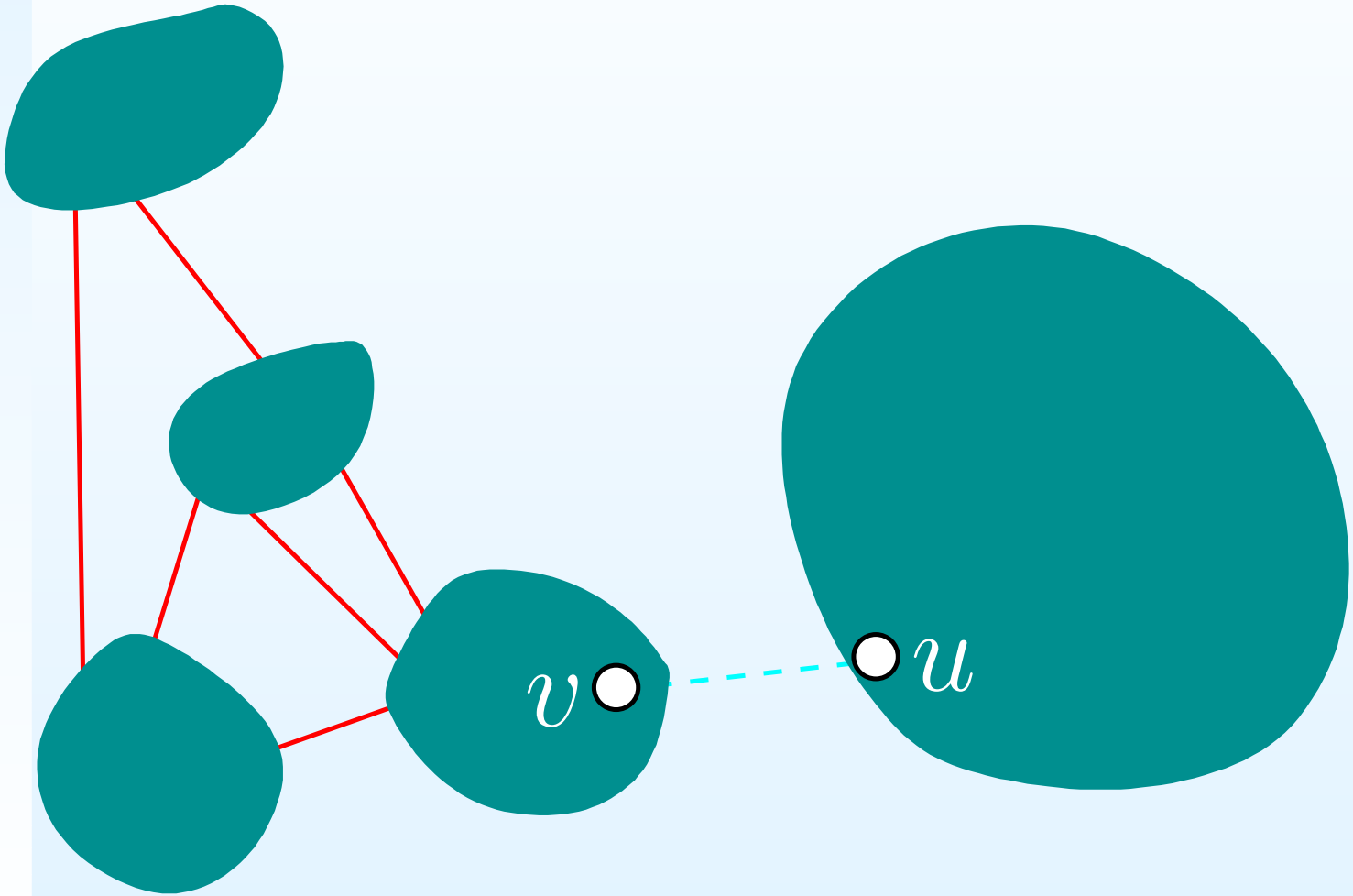
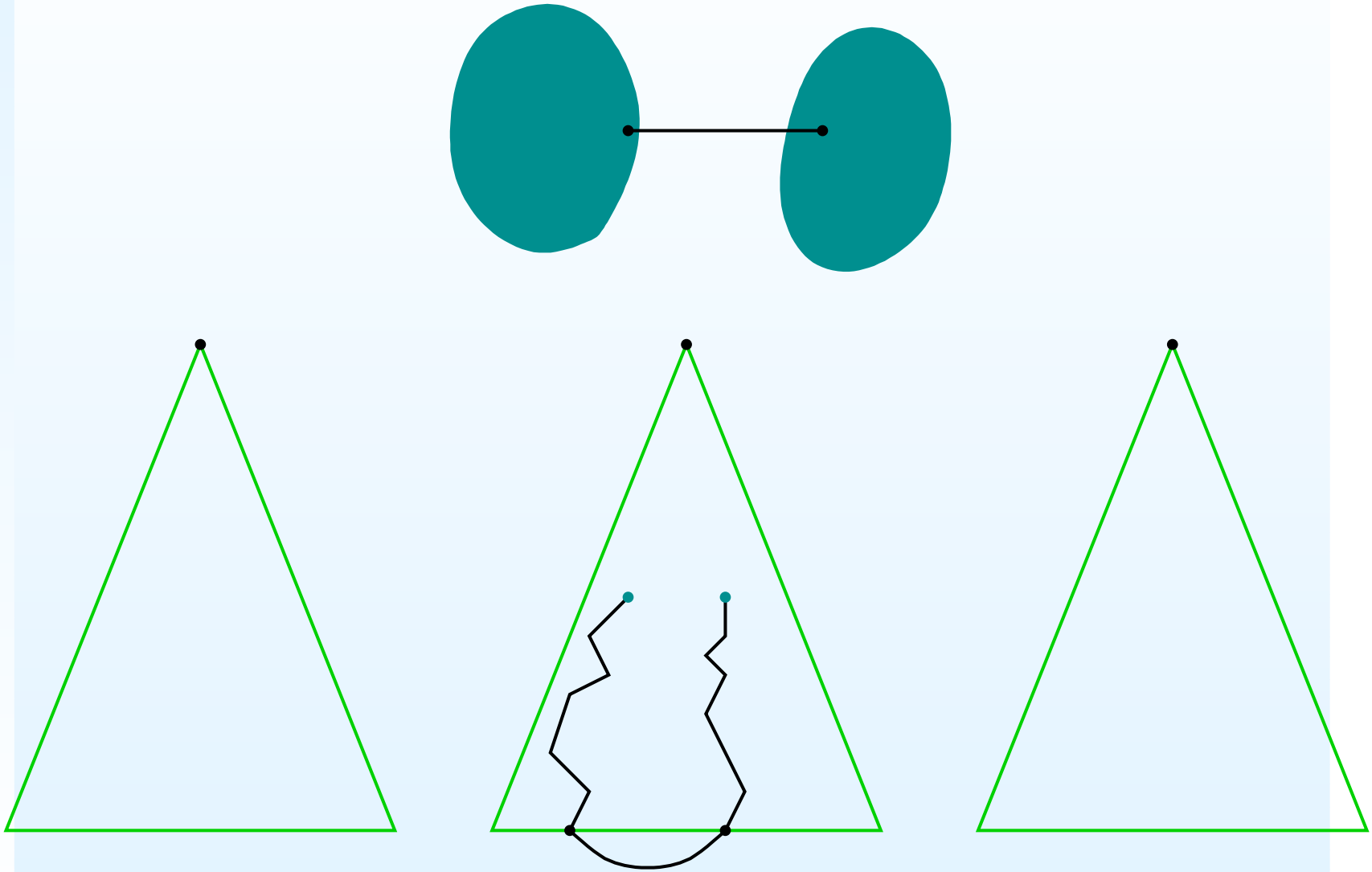


Traversing a single graph edge



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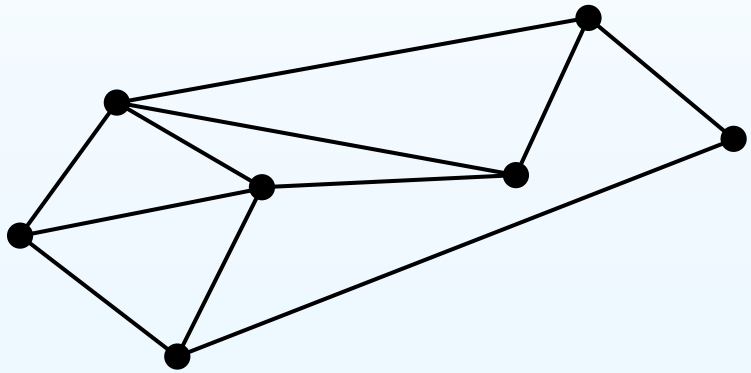

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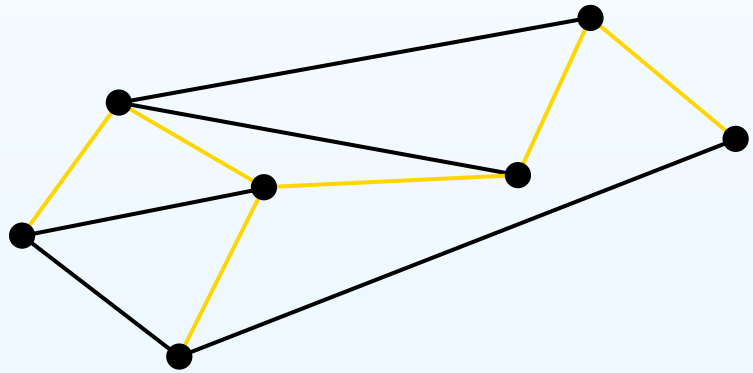

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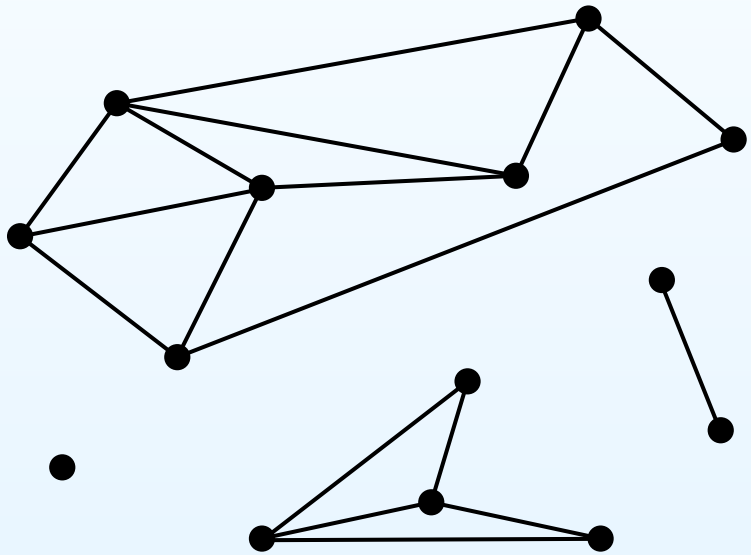

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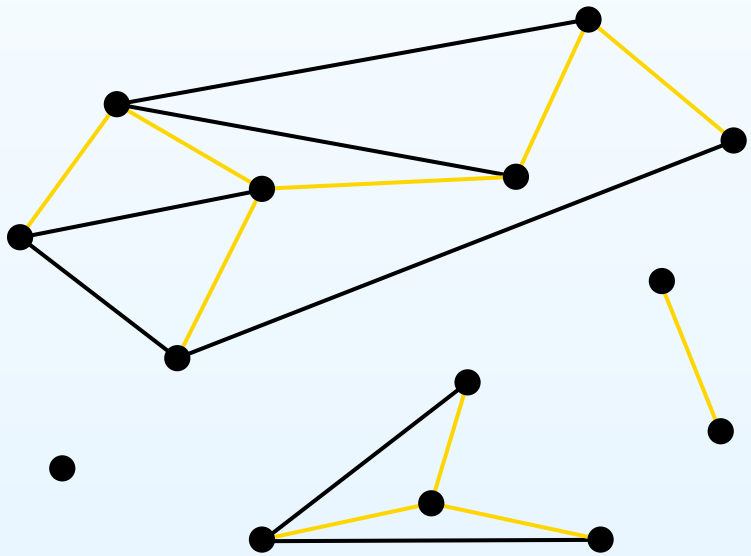

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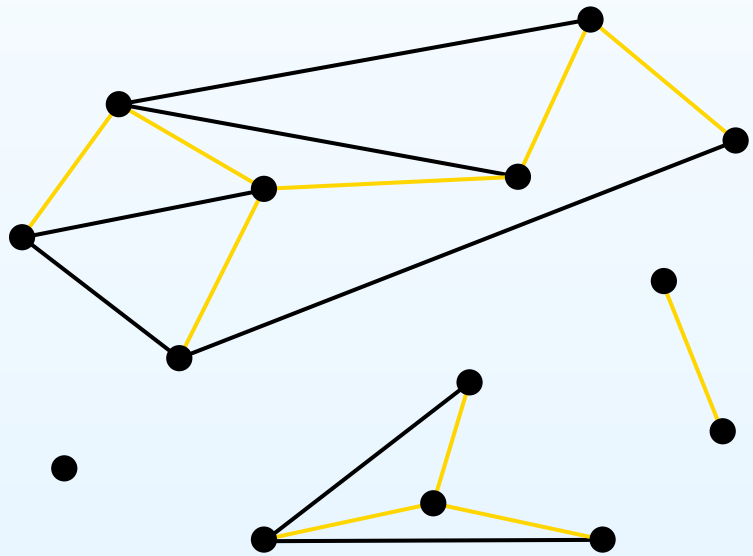
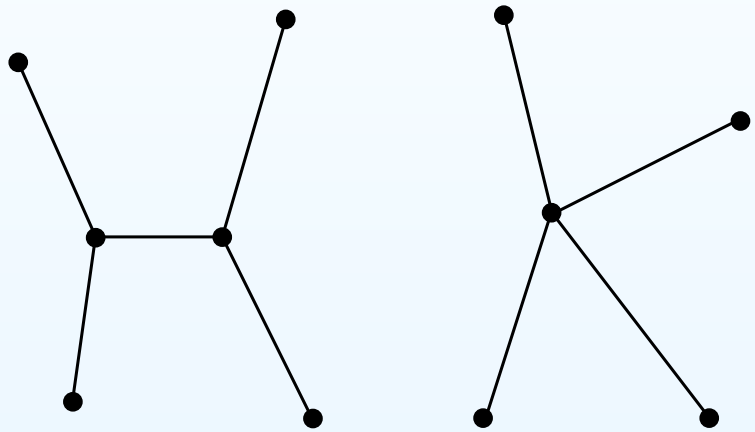


- If we can efficiently maintain a spanning forest of a dynamic graph G then we also have an efficient data structure for dynamic connectivity

Dynamic Trees

- Suppose we want a data structure maintaining some information about a dynamic forest where there is *no underlying graph*:



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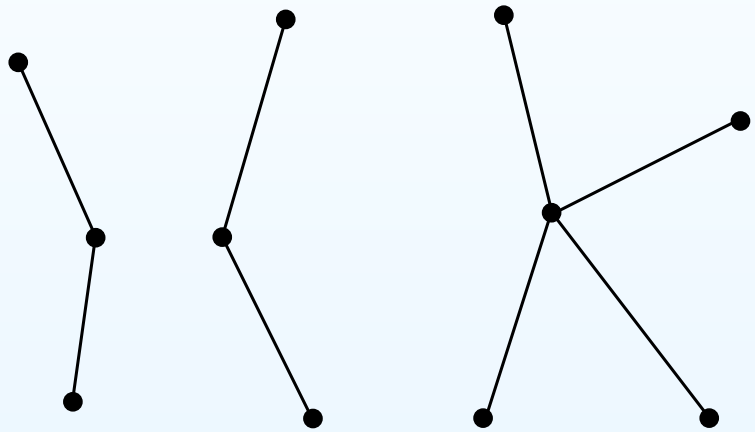

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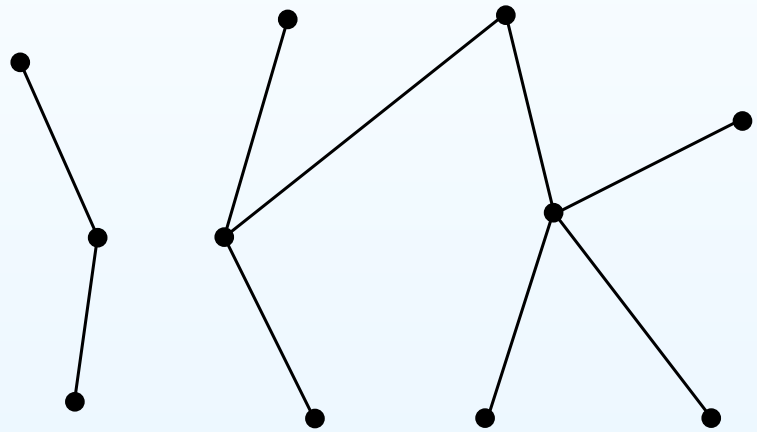

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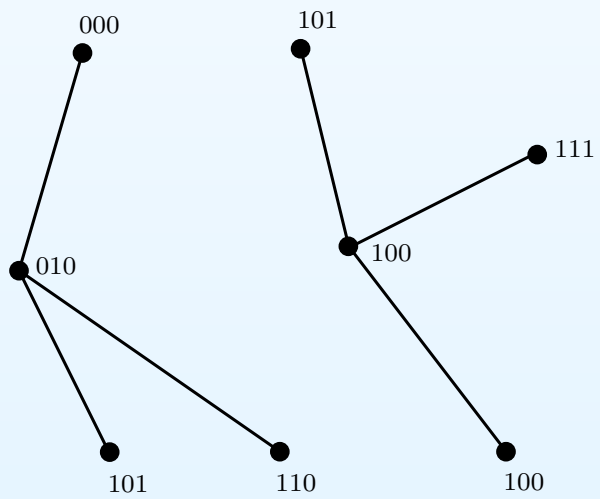

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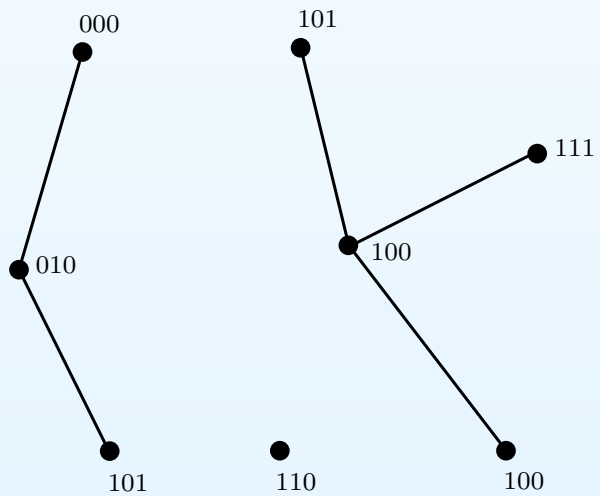

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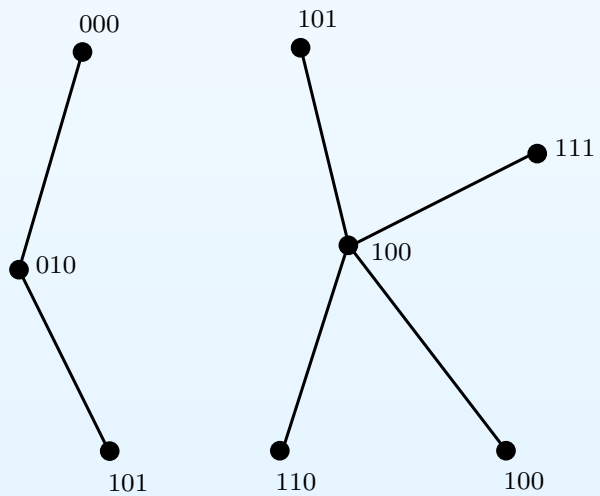

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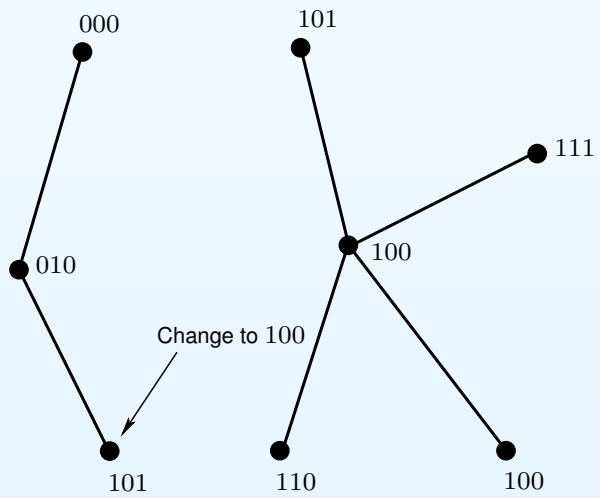

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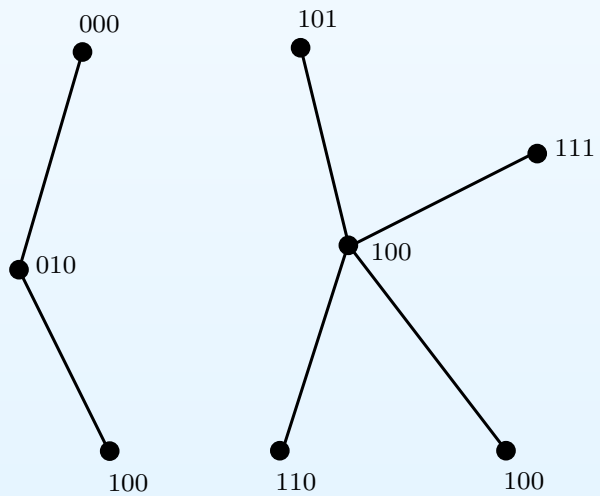

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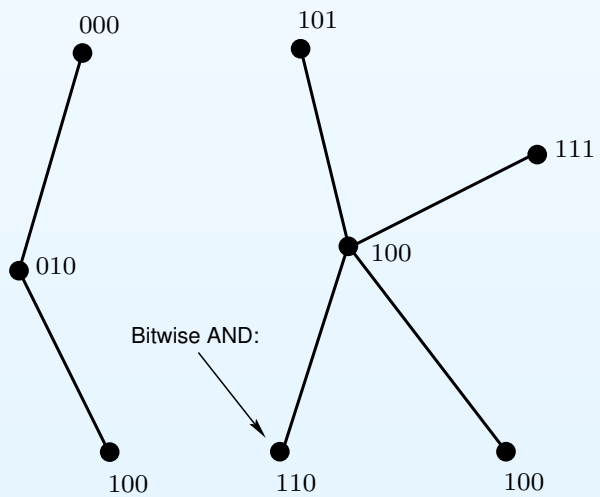

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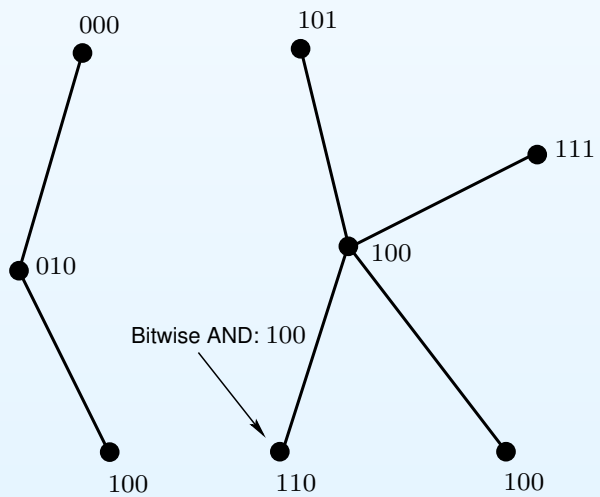

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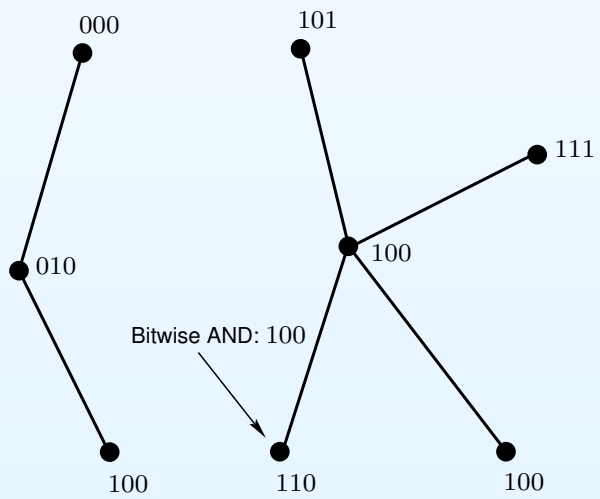
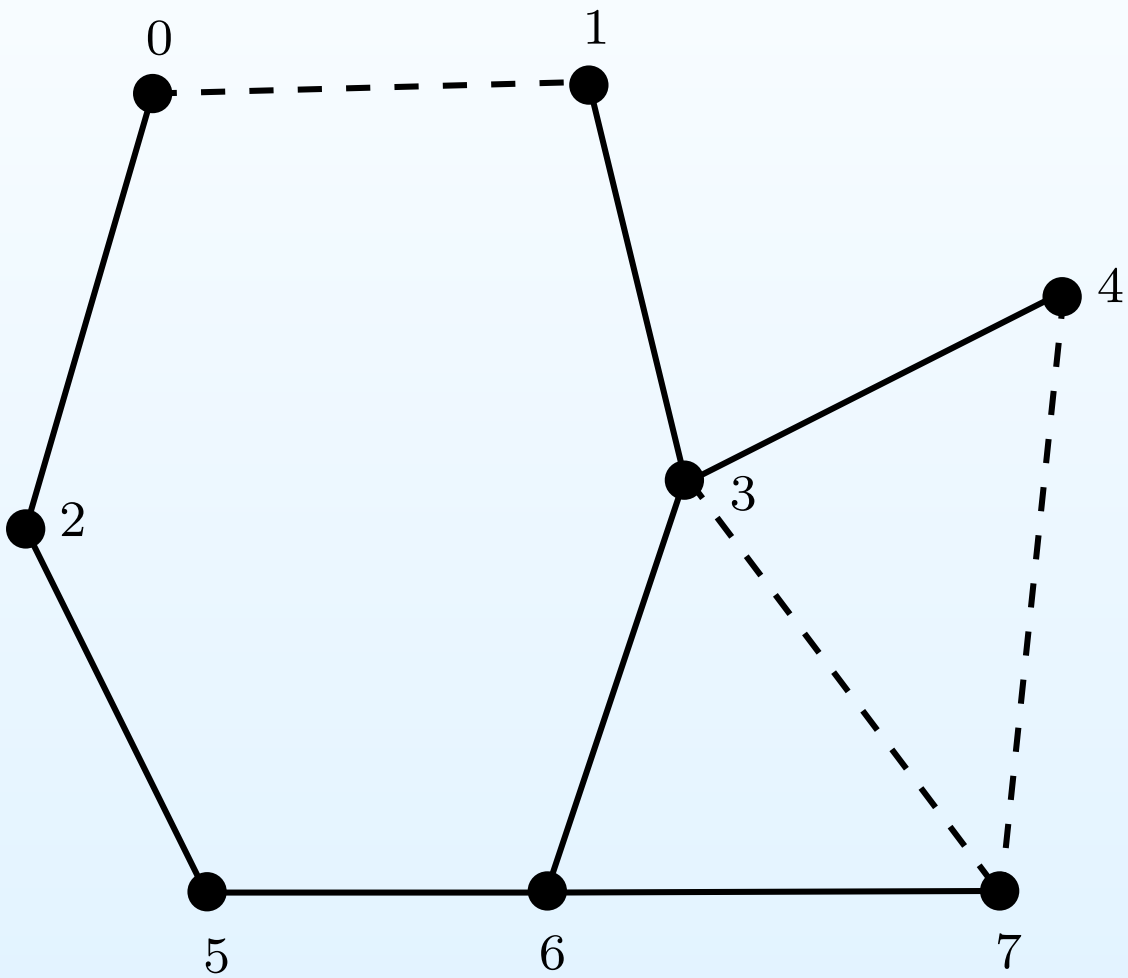


- There is a dynamic tree data structure supporting each such operation in $O(\log n)$ time

Fully-dynamic connectivity

- Efficiently maintaining a spanning forest of a graph:



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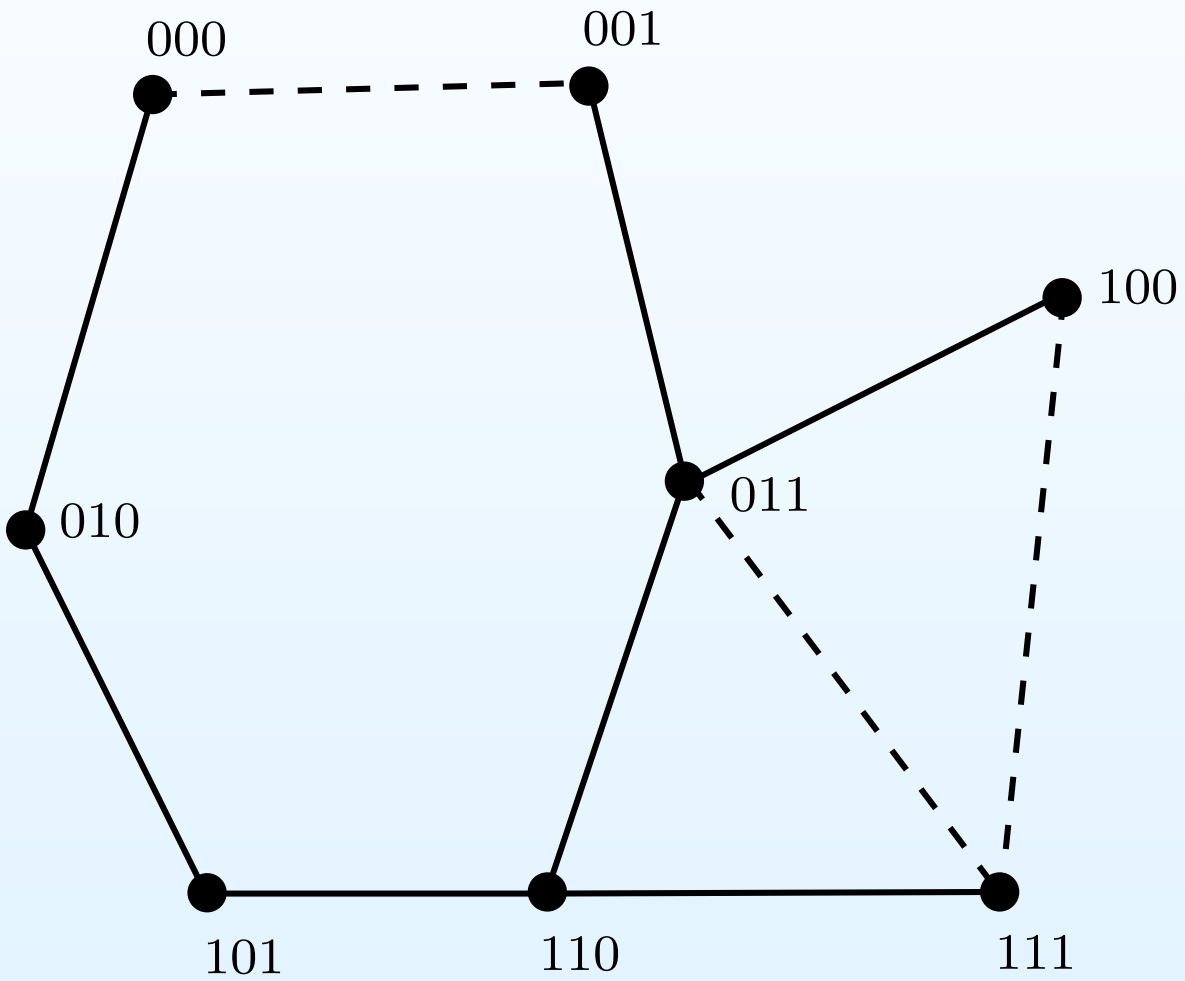

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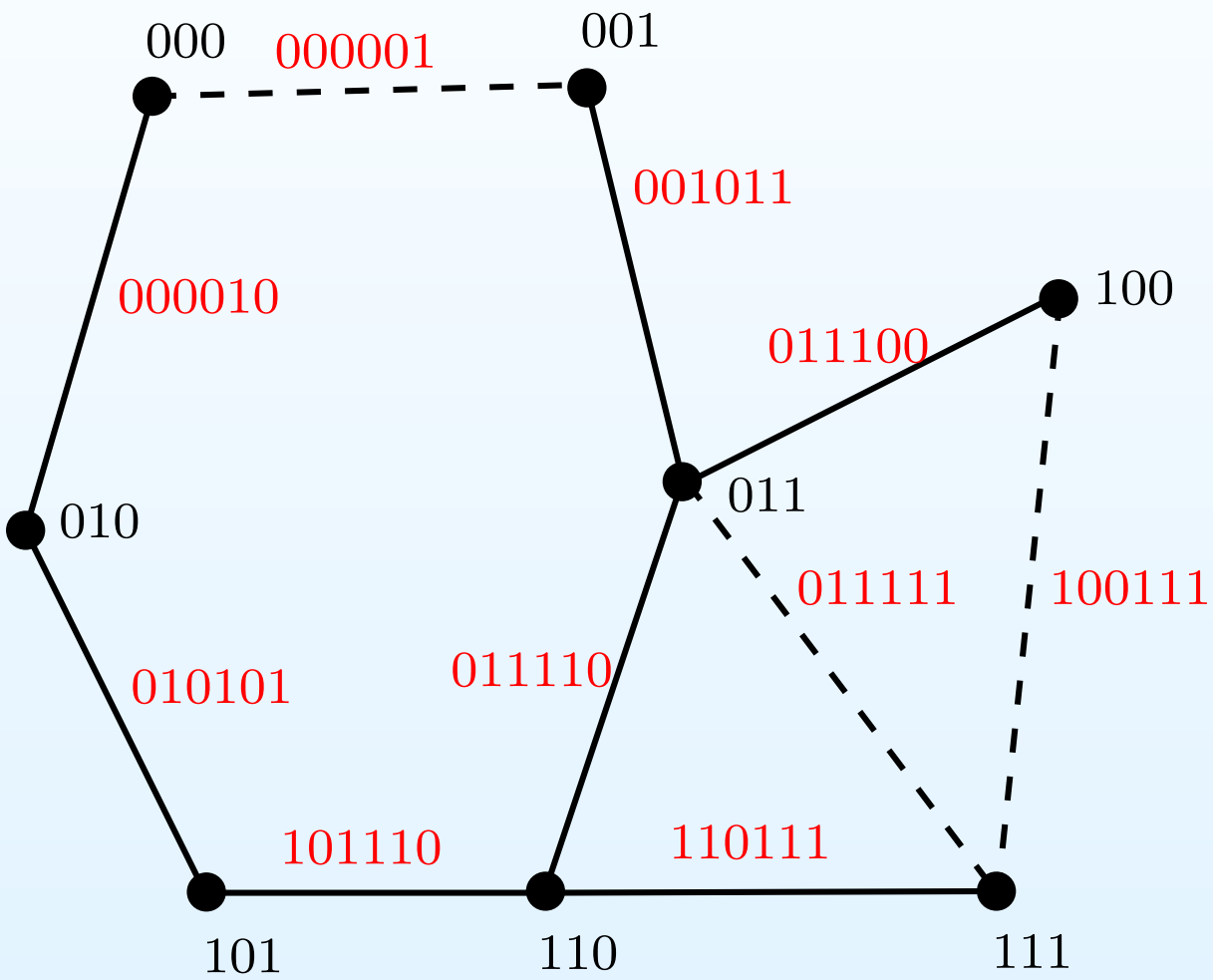

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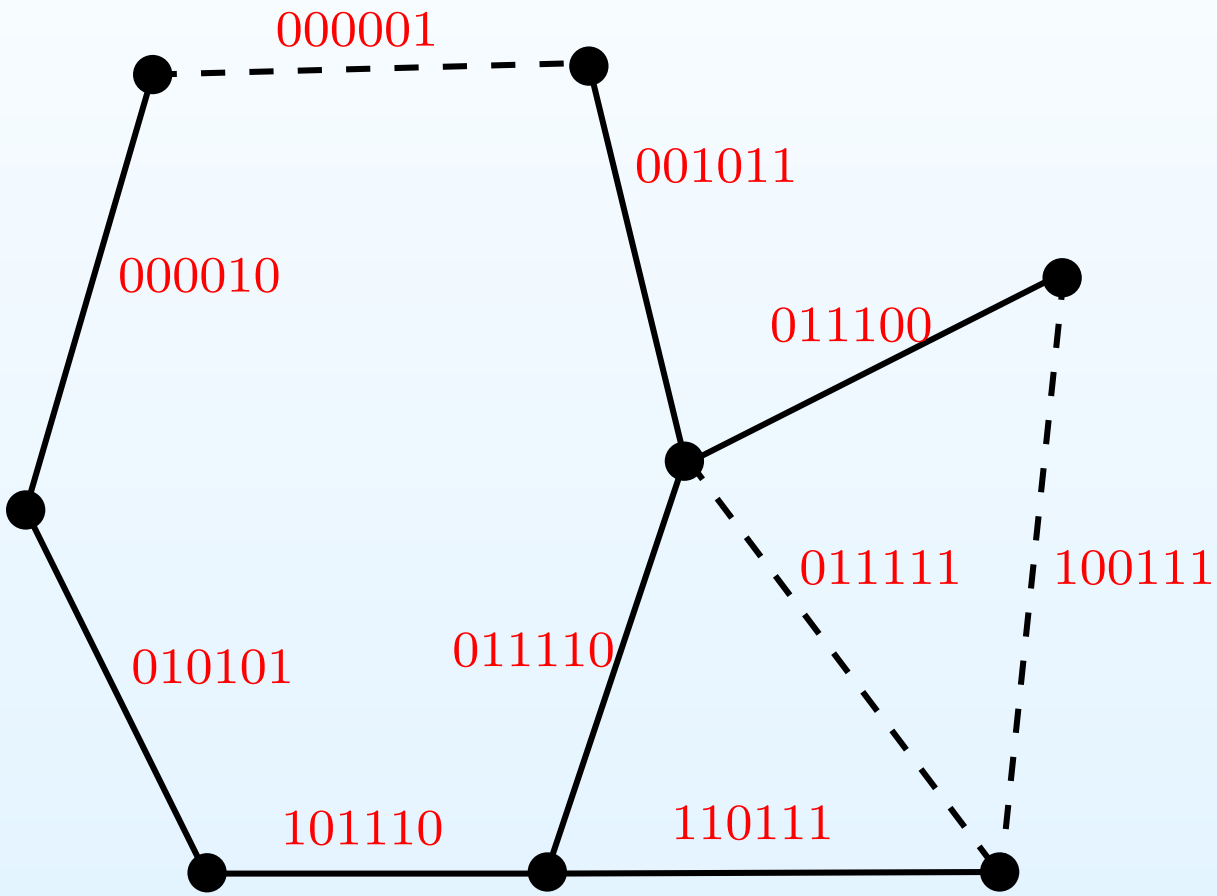

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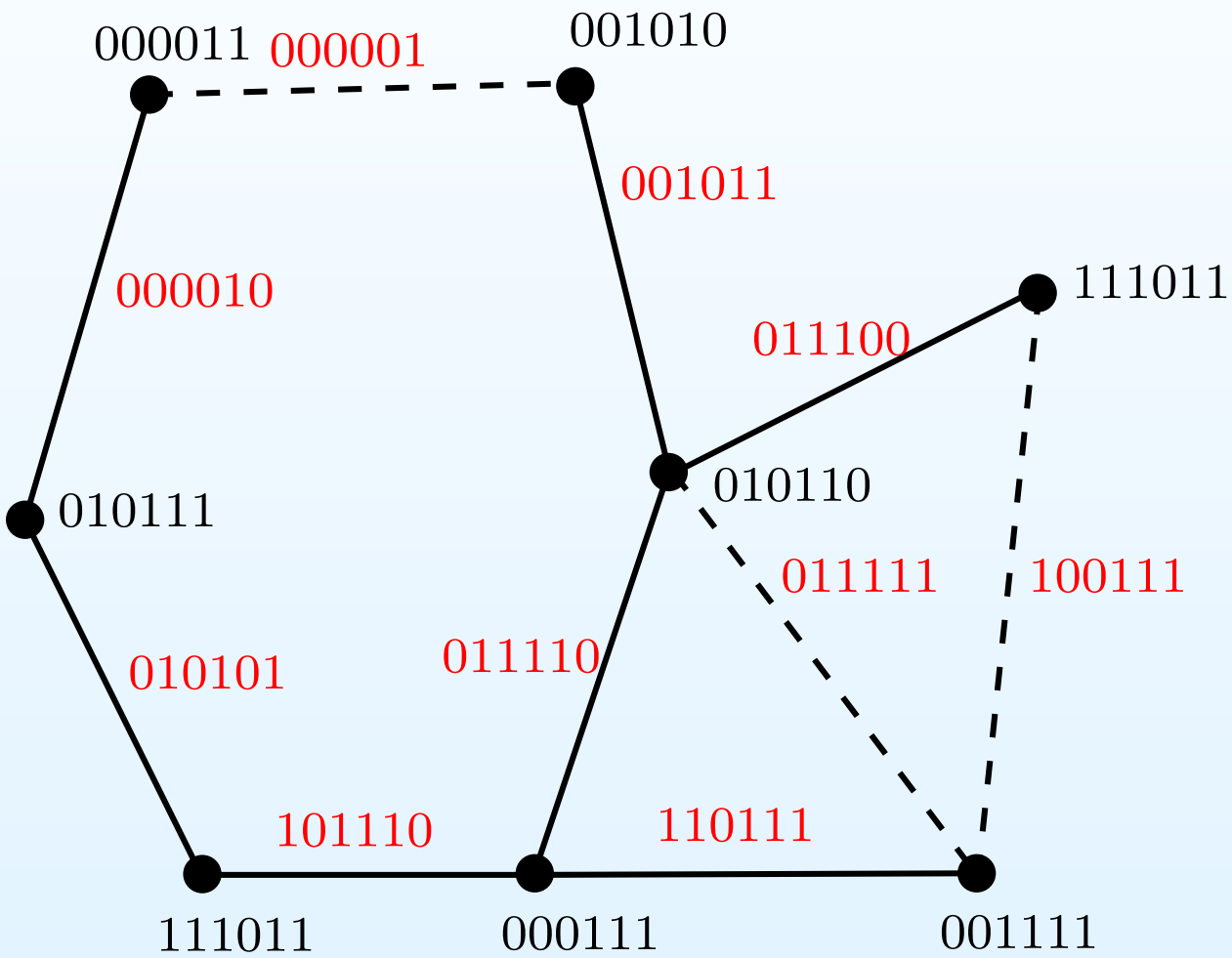

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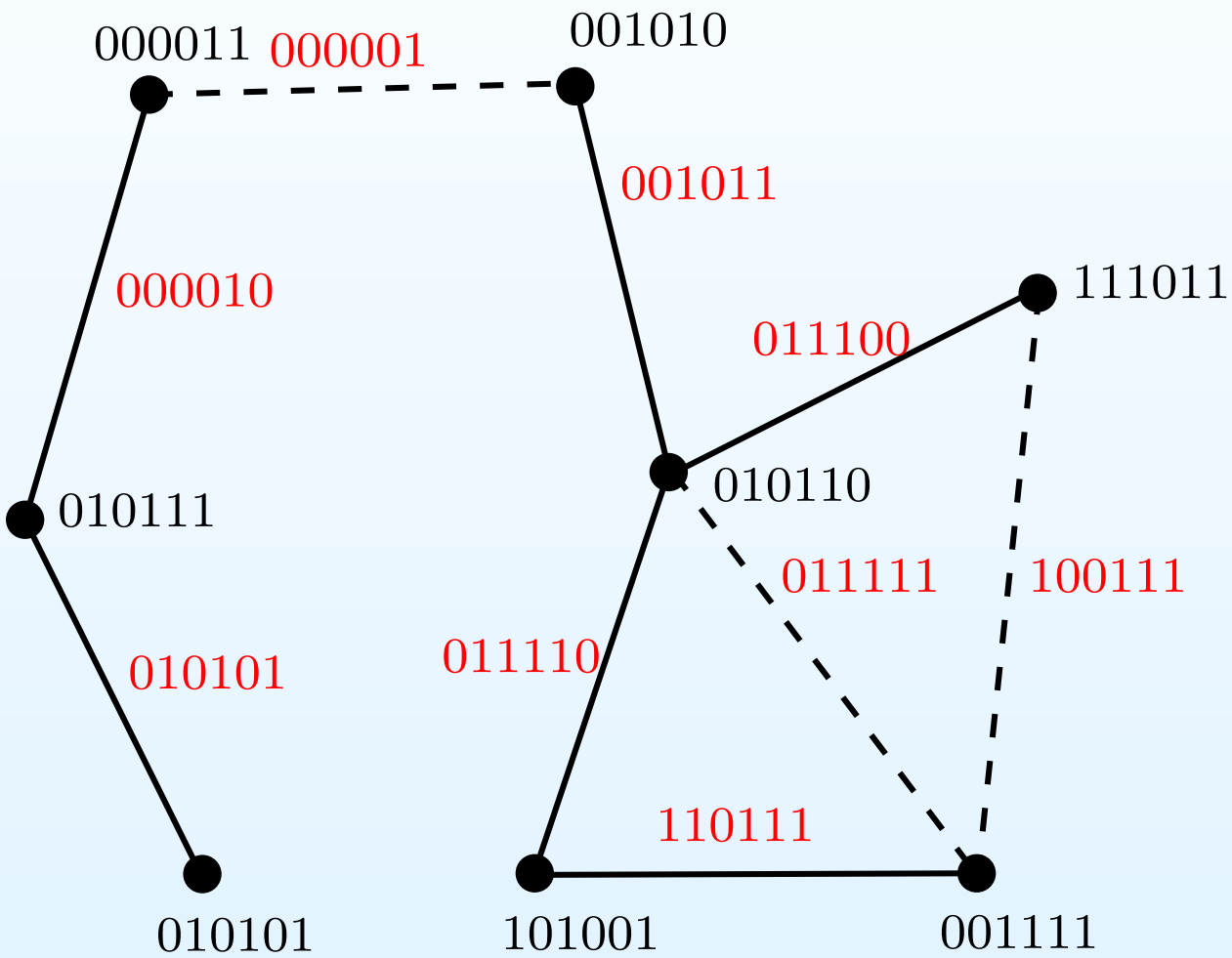

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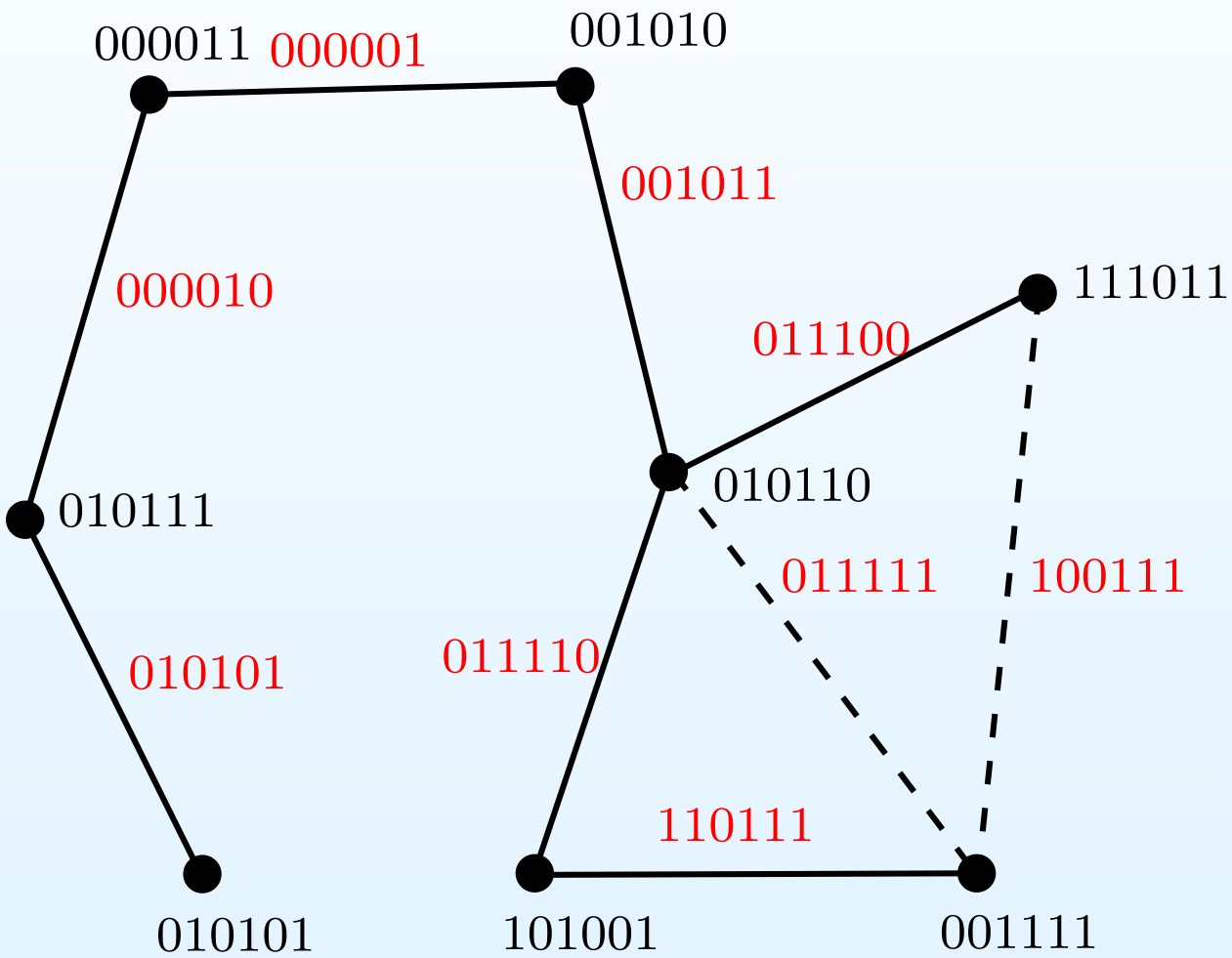

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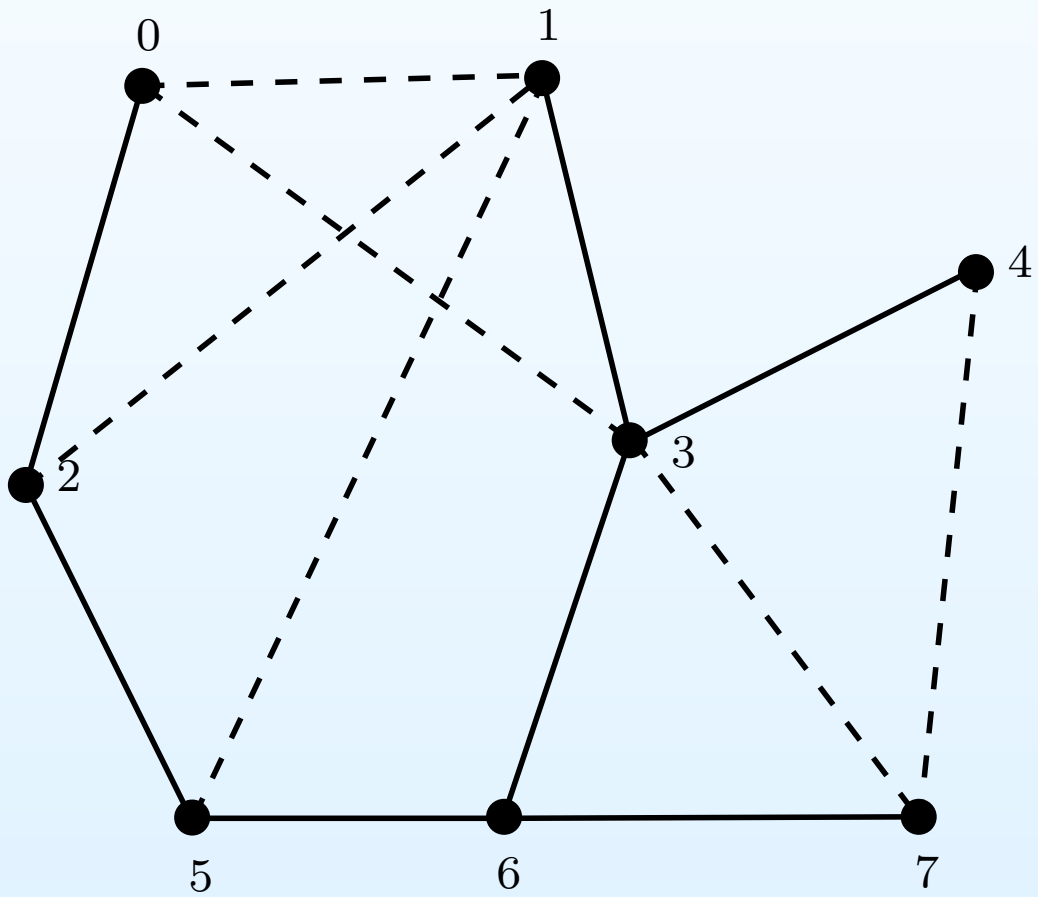

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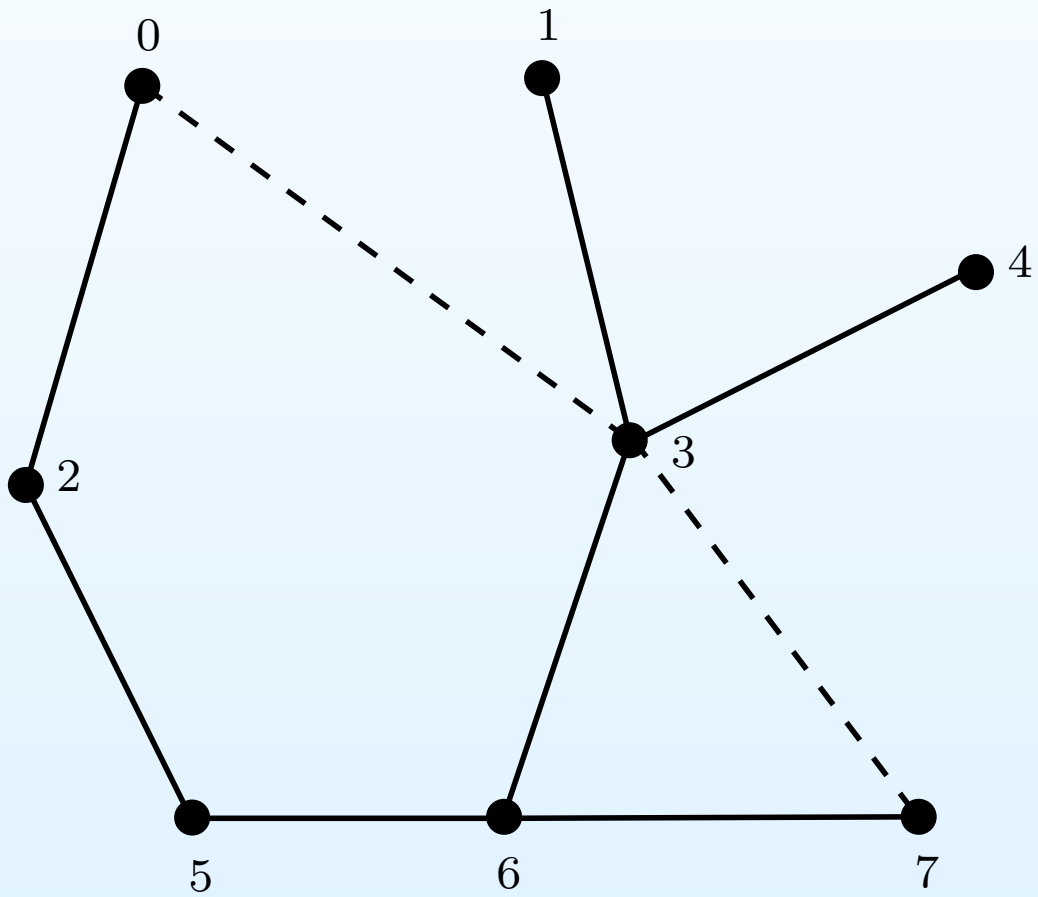

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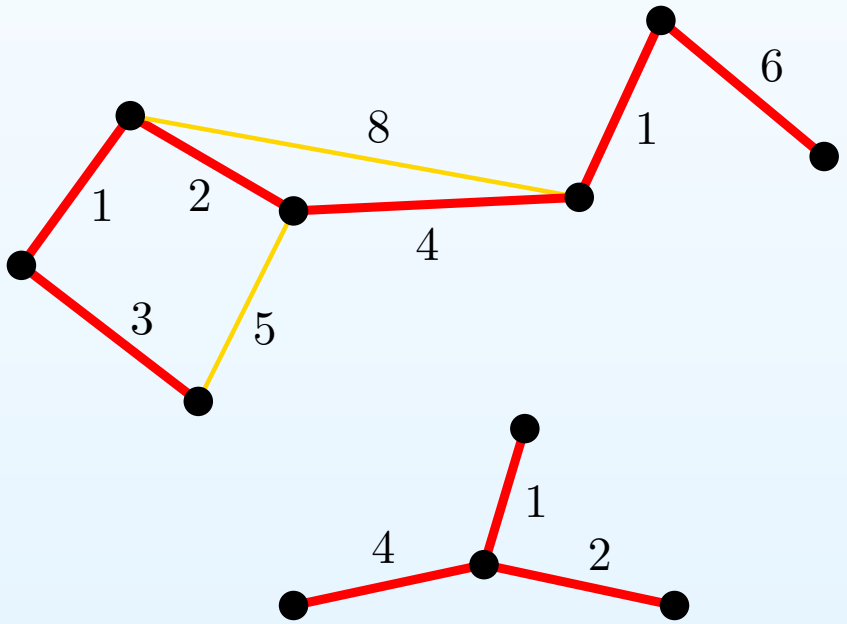

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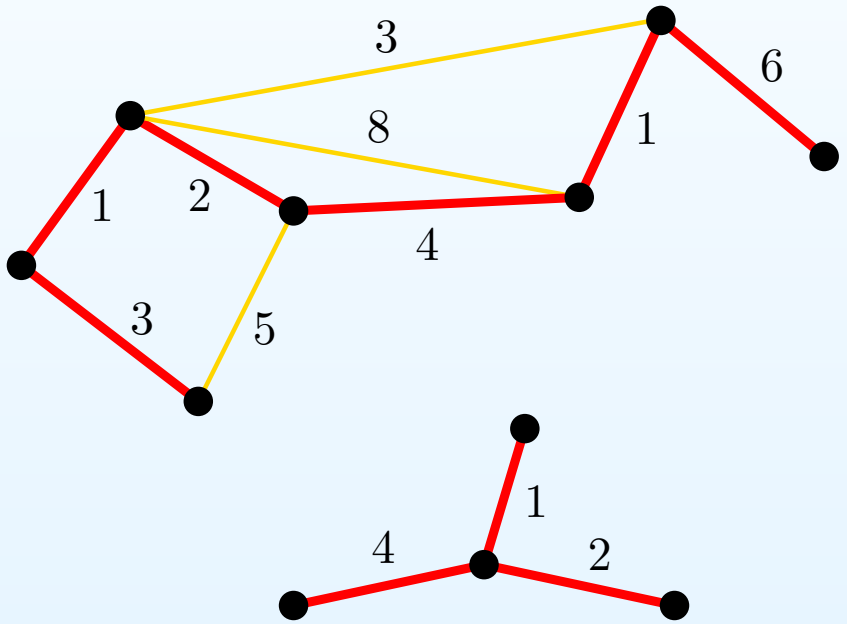

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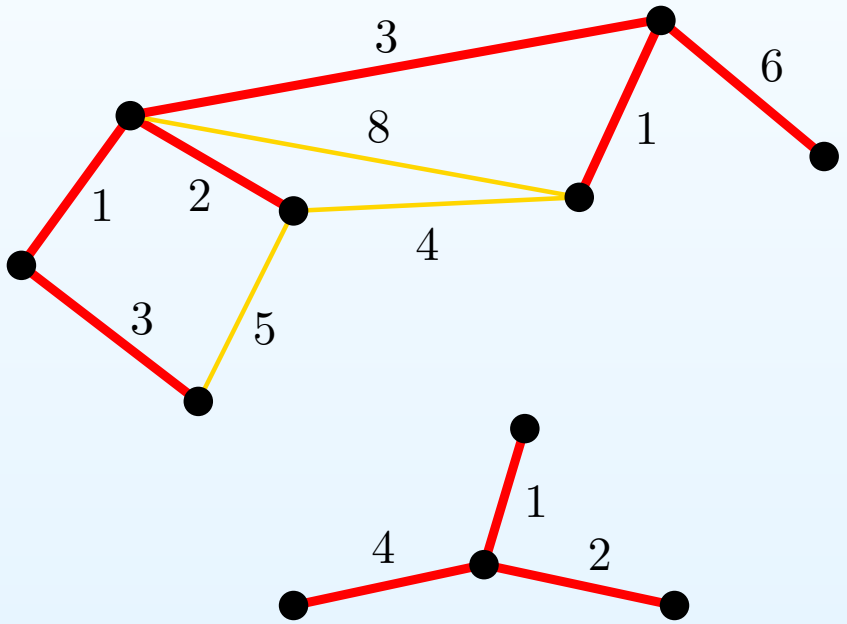

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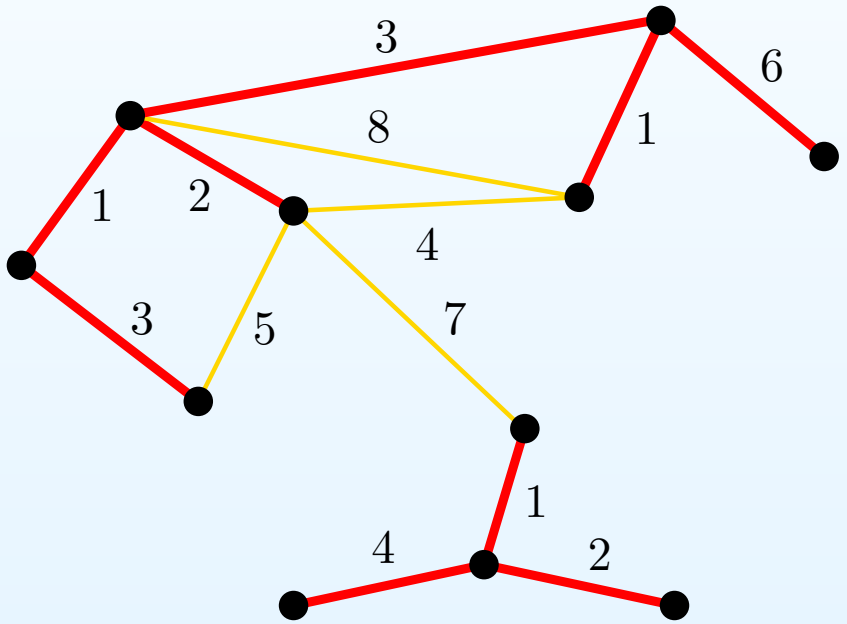

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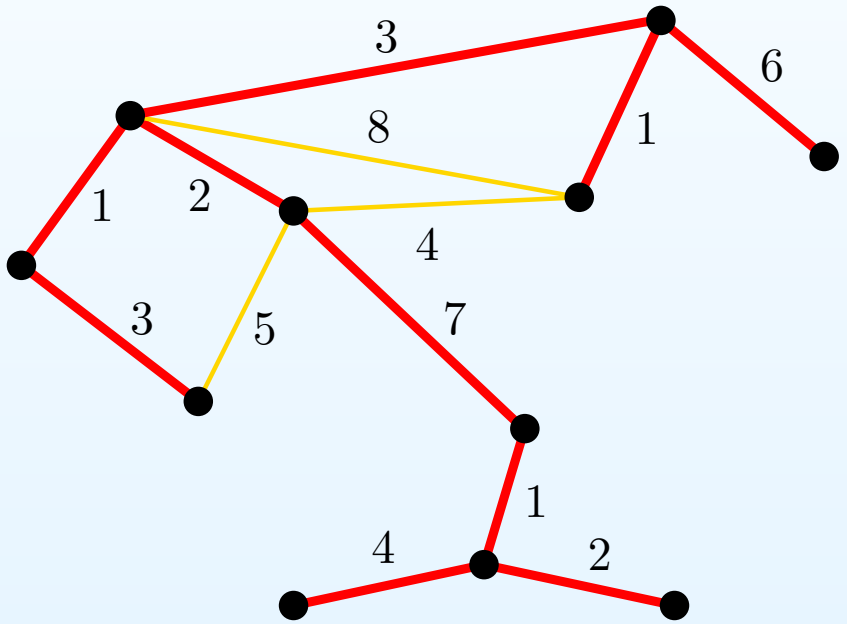

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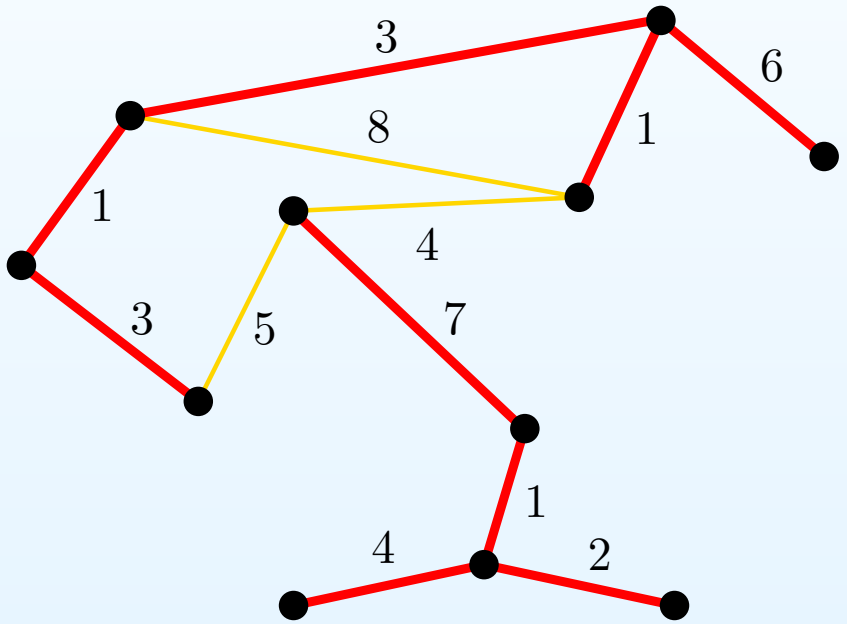

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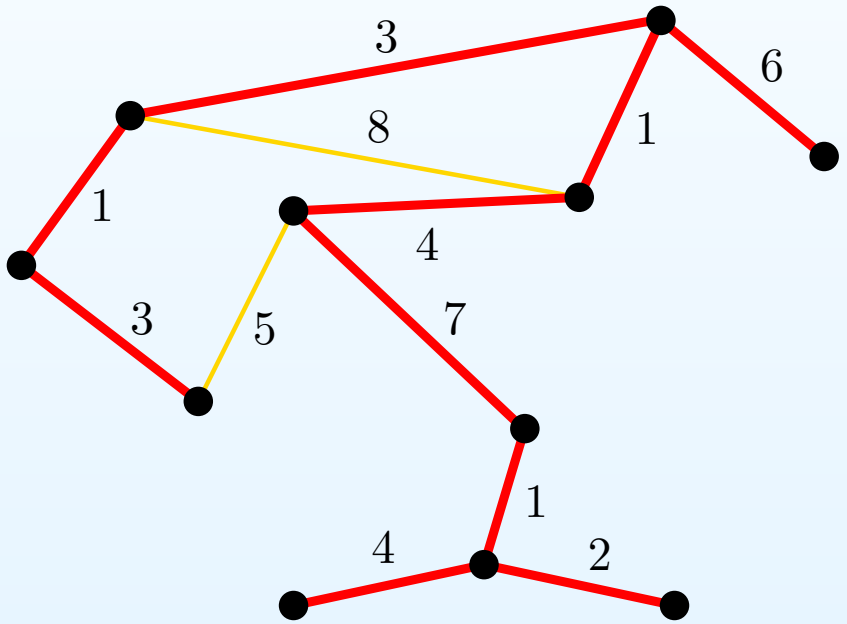

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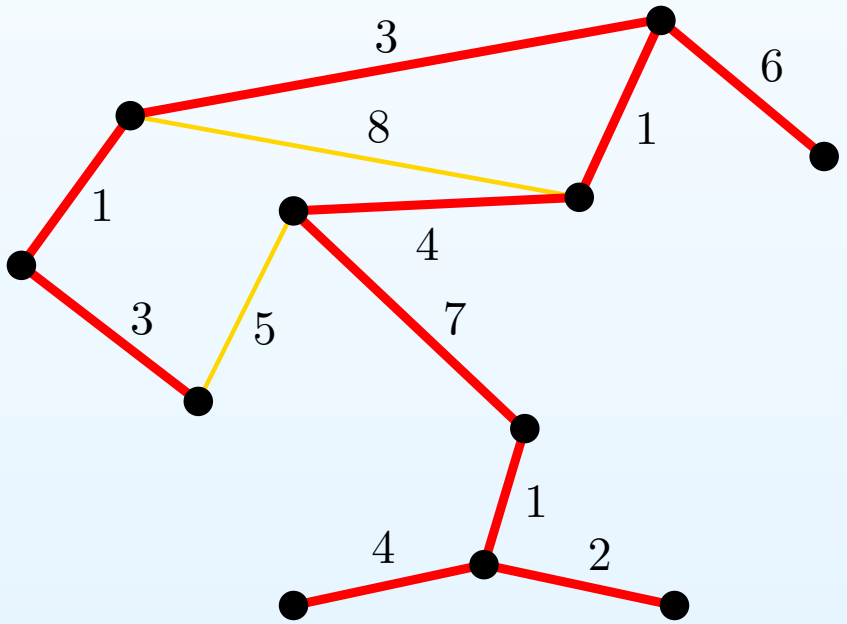

- Does the XOR trick work here?

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Issues with randomized dynamic graph algorithms

- A randomized dynamic graph algorithm either needs to:
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- Even this may potentially reveal too much information about the random bits of the algorithm

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