**PhDOpen, Warsaw**
**Symmetric Computation**
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Assignment — Due: 30 November 2019

**Note:** These exercises vary in difficulty. Some are easy consequences of material given in the lecture. Some require more thought. And for others (especially 5), it might be good to look into the published literature.

1. Given a graph $G = (V, E)$, we say that a partition of $V$ into parts $P_1, \dots, P_t$ is *equitable* if for each $i, j \in \{1, \dots, t\}$ there is a constant $\delta_{ij}$ such that each $v \in P_i$ has exactly $\delta_{ij}$ neighbours in $P_j$.

   (a) Show that there is a unique coarsest equitable partition of $V$.

   For tuples of vertices $\mathbf{u}$ and $\mathbf{v}$ of length at most $k$, we say $\mathbf{u} \equiv^{C^k} \mathbf{v}$ if there is no formula $\phi(\mathbf{x})$ of $C^k$ for which $G \models \phi[\mathbf{u}]$ and $G \not\models \phi[\mathbf{v}]$.

   (b) Show that $u \equiv^{C^2} v$ if, and only if, $u$ and $v$ are in the same part of the coarsest equitable partition of $V$.

   (c) For a pair of graphs $G$ and $H$, with coarsest equitable partitions $P_1, \dots, P_s$ and $Q_1, \dots, Q_t$ respectively, show that $G \equiv^{C^2} H$ (that is to say the two graphs are not distinguished by any sentence of $C^2$) if, and only if, $s = t$ and there is a bijection between $\{P_1, \dots, P_s\}$ and $\{Q_1, \dots, Q_t\}$ which preserves the constants $\delta_{ij}$.

   We say that a partition of $V \times V$ into parts $P_1, \dots, P_t$ is *coherent* if for each $i, j, k \in \{1, \dots, t\}$ there is a constant $\gamma_{ij}^k$ such that for each $(u, v) \in P_k$, there are exactly $\gamma_{ij}^k$ vertices $w$ such that $(u, w) \in P_i$ and $(w, v) \in P_j$.

   (d) Show that there is a unique coarsest coherent partition of $V \times V$.

   (e) Show that $(u, v) \equiv^{C^3} (u', v')$ if, and only if, $(u, v)$ and $(u', v')$ are in the same part of the coarsest coherent partition of $V \times V$.

   (f) Can you formulate a similar characterization of $\equiv^{C^k}$ for general $k$?

2. The aim of this exercise is to prove that the existence of a perfect matching in a *bipartite* graph is invariant under equivalence in $C^2$. That is to say, if two such graphs $G$ and $H$ are indistinguishable by any sentence of $C^2$, then $G$ has a perfect matching if, and only if, $H$ does.

   Let $G = (A, B, E)$ be a bipartite graph, where $E \subseteq A \times B$. Consider the coarsest equitable partition of the form $A = P_1 \cup \cdots \cup P_s$ and $B = Q_1 \cup \cdots \cup Q_t$. Let $E^+ \subseteq A \times B$ be the set of pairs $(a, b)$ such that $a \in P_i$, $b \in Q_j$ (for some $i, j$) and there is an edge between some element of $P_i$ and $Q_j$.

   (a) Show that $(A, B, E^+)$ has a perfect matching if, and only if, $(A, B, E)$ does. (*Hint:* One direction is easy. For the other direction, start with a matching $M \subseteq E^+$ and consider the weighted graph where each edge $e \in E$ between $P_i$ and $Q_j$ is assigned the weight $\frac{|M \cap (P_i \times Q_j)|}{|P_i|}$, and apply a weighted form of Hall's theorem).

(b) Show that this implies that the existence of a perfect matching is invariant under $C^2$-equivalence.

For graphs $G = (V, E)$ which are not necessarily bipartite, the property of having a perfect matching is *not* invariant under $C^2$-equivalence.

(c) Give an example of a pair of graphs $G$ and $H$ such that $G \equiv^{C^2} H$ but one has a perfect matching and the other does not.

(d) Is there a $k$ such that the class of graphs with a perfect matching is closed under $\equiv^{C^k}$? Why?

3. Show that the property of having a Hamiltonian cycle is not definable in the logic FP. (*Hint:* consider the proof given in the lecture for perfect matchings)

4. We saw in class that solvability of systems of equation over the two-element field $\mathbb{F}_2$ is not expressible in FPC. Do the same for systems of equations over the three-element field $\mathbb{F}_3$.

   Solvability of systems of equations over the rationals $\mathbb{Q}$ *is* definable in FPC. Why?

5. Prove that the class of graphs with a Hamiltonian cycle is NP-complete under first-order reductions. Explain why this implies that the class is not definable in FPC.