Game Theory: a Tutorial Introduction

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Overview

- Basic classification of the games.
- Coalitional games.
- Strategic games.
- Special case: zero-sum games.
- Extensive games.
- Special case: win or lose games.

A Basic Classification

- Cooperative (coalitional) games
  - cost allocation,
  - various settlement procedures (bankruptcy),
  - bargaining,
  - voting,
  - ...
- Non-cooperative (strategic) games
  - one-shot games (Prisoner's dilemma),
  - extensive games (chess),
  - repeated games,
    - finitely repeated games,
    - infinitely repeated games,
  - important class: auctions,
  - voting,
  - stochastic games,
  - evolutionary games,
  - ...

Handbook of Game Theory (3 vol., 1992 – 2002) has 62 chapters.

Purpose of Coalitional Games

To understand the advantages of a cooperation.

- What can be achieved through cooperation?
- How profits should be shared (axiomatic approach)?
- What can be achieved through negotation and bargaining?

Example: Sharing a water supply system

A company considers building a water supply system that is to be shared between three villages. The costs depend on for whom the system is to be built:

$$c(1) = 12,\ c(2) = 15,\ c(3) = 12,$$
$$c(12) = 19,\ c(13) = 20,\ c(23) = 21,$$
$$c(123) := 26.$$

Question How to divide the costs between the villages?

- Core of a coalitional game: set of allocations that are
  - efficient
    (the costs add up to the cost paid by the grand coalition),
  - group rational
    (no group will get a better deal by breaking away).
- In the above example
  - $c(1) = 8,\ c(2) = 10,\ c(3) = 8$ is in the core, while
  - $c(1) = 10,\ c(2) = 10,\ c(3) = 6$ is not.

Coalitional Games: Definition

Coalitional game for $n \geq 2$ players:

- $N = \{1, 2, \ldots, n\}$ is called the grand coalition.
- A coalitional game: (N, v), where $v : \mathcal{P}(N) \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$.
- Intuition: $v(C)$ is the value (worth) that can be achieved by coalition C.
- Given a cost sharing game with cost function c:

$$v(C) = \sum_{i \in C} c(\{i\}) - c(C).$$

- Efficient allocation:

$$f : N \rightarrow \mathbb{R},$$

  such that $\sum_{i \in N} f(i) = v(N)$.
  Abbreviation: $f(S) = \sum_{i \in S} f(i)$.
- Group rationality:
  $f(S) \geq v(S)$ for all coalitions S of N.
- Core of a coalitional game:

$$Core(v) := \{f \in \mathbb{R}^n \mid f \text{ is an efficient allocation that satisfies group rationality}\}.$$

Shapley Value

- Allocation function assigns to each coalitional game an allocation.
- Shapley Value is an allocation function ϕ such that:

symmetry: for all $i, j \in N$,
if for all $C \subseteq N \setminus \{i,j\}$, $v(C \cup \{i\}) = v(C \cup \{j\})$, then $\phi_i(v) = \phi_j(v)$,

dummy: for all $i \in N$,
if for all $C \subseteq N \setminus \{i\}$, $v(C) = v(C \cup \{i\})$, then $\phi_i(v) = 0$,

additivity: $\phi(u+v) = \phi(u) + \phi(v)$, where $(N, u+v)$ is the sum of the games (N, u) and (N, v).

- A value function ϕ is a function that assigns to each cooperative game (N, v) an efficient allocation.

Theorem (Shapley, 1953) The Shapley value is a unique value function ϕ that satisfies the above three axioms.

Purpose of Strategic Games

To understand the consequences of a competition.

- How to model a competition?
- What outcomes can a game have assuming the players are rational?
- What are the best outcomes for a society?
- In two-players zero-sum games: what are the winning strategies?

Example: Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Question Which decisions (strategies) will choose the players?

- Best response: a strategy that yields the best payoff against the strategies chosen by the other players.
- Nash equilibrium: a joint strategy in which each player selected a best response.
- Social optimum: a joint strategy that maximizes the aggregate payoff.

- In the Prisoner's Dilemma game
  - (D, D) is a (unique) Nash equilibrium, while
  - (C, C) is a (unique) social optimum.

Strategic Games: Definition

Strategic game for $n \geq 2$ players:

- (possibly infinite) set S_i of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \to \mathbb{R}$,

for each player i.

Basic assumptions:

- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others' rationality.

Main Concepts

- Notation: $s_i, s'_i \in S_i$, $s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n$.
- s_i is a best response to s_{-i} if

$$\forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- s is a Nash equilibrium if $\forall i \;\; s_i$ is a best response to s_{-i}:

$$\forall i \in \{1, \ldots, n\} \; \forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- Intuition: In a Nash equilibrium no player can gain by *unilaterally* switching to another strategy.
- Social welfare of s: $SW(s) = \sum_{j=1}^{n} p_j(s)$.
- s is a social optimum if it is a maximum of $SW(\cdot)$.

Example: Prisoner's Dilemma for n Players

- $n > 1$ players,
- two strategies:
  1 (formerly C),
  0 (formerly D).

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

- For $n = 2$ we get the original Prisoner's Dilemma game.
- $\sum_{j \neq i} s_j$ equals the number of 1 strategies in s_{-i}.
- Let $\mathbf{1} = (1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$.
- $\mathbf{0}$ is the unique Nash equilibrium, with social welfare n.
- $\mathbf{1}$ is the unique social optimum, with social welfare $2n(n-1)$.

Nash Equilibria in General

- They do exist for many types of games.
- Important examples:
  - congestion games,
  - finite extensive games.
- In general Nash equilibria do not need to exist.
- Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Nash Theorem

- Given a finite strategic game G.
- A mixed extension of G:
  - strategies: mixed strategies.
    A mixed strategy: a probability distribution over the original (pure) strategies.
  - payoffs: canonic extensions of the original payoff functions to mixed strategies.
- Nash Theorem (1950): Every mixed extension a finite strategic game has a Nash equilibrium.

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, \quad 1$
T	$-1, \quad 1$	$1, -1$

- Mixed strategies: $\alpha H + (1-\alpha)T$, where $\alpha \in [0, 1]$.
- Payoff functions:
  Let $m_1 = \alpha_1 H + (1-\alpha_1)T$, $m_2 = \alpha_2 H + (1-\alpha_2)T$.

$$\begin{aligned} p_1(m_1, m_2) = & \quad \alpha_1\alpha_2 p_1(H, H) + \alpha_1(1-\alpha_2)p_1(H, T) \\ & + (1-\alpha_1)\alpha_2 p_1(T, H) + (1-\alpha_1)(1-\alpha_2)p_1(T, T) \\ = & \; \alpha_1\alpha_2 - \alpha_1(1-\alpha_2) - (1-\alpha_1)\alpha_2 + (1-\alpha_1)(1-\alpha_2) \\ = & \; 4\alpha_1\alpha_2 - 2(\alpha_1 + \alpha_2) + 1. \end{aligned}$$

$$p_2(m_1, m_2) = -p_1(m_1, m_2).$$

- Nash equilibrium:
  $m_1 = 0.5H + 0.5T$, $m_2 = 0.5H + 0.5T$.
- Payoffs in Nash equilibrium to both players: 0.

Minimax Theorem of Von Neumann

- Zero-sum game:
  a two-players game in which the sum of the payoffs is 0.
- Minimax Theorem (Von Neumann, 1928)
  Consider a finite zero-sum game $G := (S_1, S_2, p_1, p_2)$. Then for $i = 1, 2$

$$\min_{m_{-i} \in M_{-i}} \max_{m_i \in M_i} p_i(m_i, m_{-i}) = \max_{m_i \in M_i} \min_{m_{-i} \in M_{-i}} p_i(m_i, m_{-i}).$$

- This theorem implies that in zero-sum games the mixed strategies in Nash equilibria are of a certain form (pairs of security strategies).
- Informally. A security strategy guarantees a best outcome against a 'mean' opponent.
- The payoffs in all Nash equilibria are the same.

Historical Remarks

- First special case of Nash theorem: Cournot (1838).
- Nash theorem (1950) generalizes one aspect of von Neumann's Minimax Theorem (1928).
- Nash proved his theorem using Kakutani's Fixed Point Theorem.
- An alternative proof (also by Nash) uses Brouwer's Fixed Point Theorem.

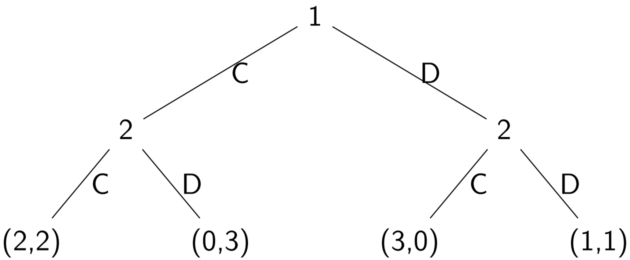
Extensive Games

Example: Prisoner's Dilemma

Strategic game:

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Extensive game:



Discussion

- This is an example of a two-player games with two stages.
- In general there may be more players and more stages.
- We limit ourselves to the games with finitely many stages (games with finite horizon) and such that at each stage exactly one player proceeds.
- At each stage a player can have infinitely many choices.
- We assume here perfect information: each player knows the previous moves.

Extensive Game: Definition

Extensive game for $n \geq 1$ players:

- game tree: a finite depth tree $T := (V, E)$ with a turn function $D : V \setminus Z \to \{1, \ldots, n\}$, where Z is the set of leaves of T,
- outcome function $o_i : Z \to \mathbb{R}$, for each player i.

We denote it by $(T, D, o_1, \ldots, o_n)$.

- Given $v \in V \setminus Z$ we call $\{w \mid (v, w) \in E\}$ the set of actions available to player $D(v)$ at node v.
- Sometimes we identify the actions with the labels put on the edges.

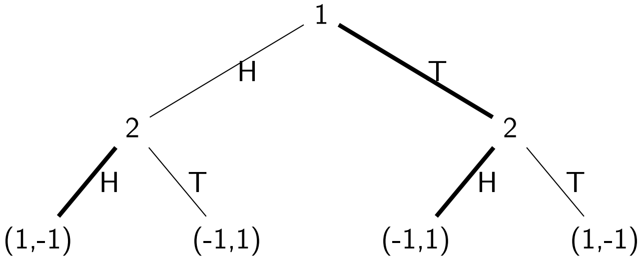
Strategies

Consider an extensive game $EG := (T, D, o_1, \ldots, o_n)$.

- Let $N_i := \{v \in V \mid D(v) = i\}$.
  N_i is the set of nodes at which player i takes an action.
- Strategy for player i:
  $s_i : N_i \to V$, such that for all $v \in N_i$, $(v, s_i(v)) \in E$.
- Joint strategy: $s = (s_1, \ldots, s_n)$.
  It assigns a unique edge to every node in $V \setminus Z$.
- To each joint strategy s there corresponds a finite path $path(s) := (v_1, \ldots, v_h)$ in T defined inductively:
  - v_1 is the root of T,
  - if $v_k \notin Z$, then $v_{k+1} := s_i(v_k)$, where $D(v_k) = i$.
- When each player i selects s_i we call $(o_1(z), \ldots, o_n(z))$, where z is the last element of $path(s)$, the outcome of EG.

Example of Strategies: Matching Pennies

	H	T
H	$1, -1$	$-1, \ 1$
T	$-1, \ 1$	$1, -1$



Strategies for player 1: H, T.
Strategies for player 2: HH, HT, TH, TT.
Thick lines correspond with the joint strategy (T,HH).

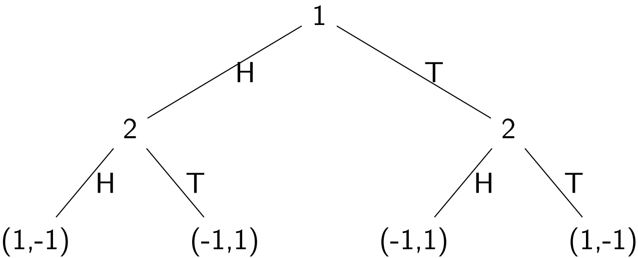
Strategic Forms

With each extensive game $EG := (T, D, o_1, \ldots, o_n)$ we associate a strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ defined as follows:

- S_i is the set of strategies of player i in EG,
- $p_i(s) := o_i(z)$, where z is the last element of $path(s)$.
- G is called the strategic form of EG.
- s is called a Nash equilibrium of EG if it is a Nash equilibrium of G.

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, \quad 1$
T	$-1, \quad 1$	$1, -1$



Strategic form

	HH	HT	TH	TT
H	$1, -1$	$1, -1$	$-1, \quad 1$	$-1, \quad 1$
T	$-1, \quad 1$	$1, -1$	$-1, \quad 1$	$1, -1$

Note. Two Nash equilibria: (H, TH) and (T, TH).

Subgames

Consider $EG := (T, D, o_1, \ldots, o_n)$.

- We define the subgame of EG rooted at node v of T, EG^v, as expected.
- Note. Each strategy s_i of player i in EG uniquely determines his strategy s_i^v in EG^v.
- $(s_1, \ldots, s_n)$ is called a subgame perfect equilibrium in EG if for each node v of T $(s_1^v, \ldots, s_n^v)$ is a Nash equilibrium in EG^v.
- Informally: s is subgame perfect equilibrium in EG if it induces a Nash equilibrium in every subgame of EG.

Backward Induction

- Given a tree (V, E) and $v \in V$, let $desc(v) := \{w \mid (v, w) \in E\}$.
- Fix a finite extensive game $EG := ((V, E), D, o_1, \ldots, o_n)$.
- Backward induction algorithm

  while $|V| > 1$ **do**
      choose $v \in V$ such that all its descendants are leaves;
      $i := D(v)$;
      choose $w \in desc(v)$ such that $o_i(w)$ is maximal;
      $s_i(v) := w$;
      for $j \in \{1, \ldots, n\}$ **do** $o_j(v) := o_j(w)$ **od**;
      $V := V \setminus desc(v)$; $E := E \cap (V \times V)$;
  od

- Note. This process generates a set of joint strategies. Multiple joint strategies may arise due to the second **choose** statement.

Kuhn and Selten Theorems

- Theorem (Kuhn, 1950)
  Every finite extensive game (with perfect information) has a Nash equilibrium.
- Theorem (Selten, 1965)
  Every finite extensive game (with perfect information) has a subgame perfect equilibrium.
- A stronger claim holds:
  A joint strategy is a subgame perfect equilibrium iff it can be generated by the backward induction algorithm.

Win or Lose Games

- A two-player extensive game is called a win or lose game if the only possible outcomes are $(1, -1)$ and $(-1, 1)$.
- Notation: $-i$ is the opponent of player i.
- s_i is called a winning strategy of player i in a win or lose game EG if

$$\forall s_{-i} \in S_{-i} p_i(s_i, s_{-i}) = 1,$$

  where (S_1, S_2, p_1, p_2) is the strategic form of EG.
- Theorem (Zermelo, 1913)
  In every win or lose game one of the players has a winning strategy.

Proof of Zermelo's Theorem

Theorem In every win or lose game one of the players has a winning strategy.

- We can assume that the players alternate their moves.
- We can extend all the paths in the game so that all paths in T are of the same depth, say $2k$.
- Let W denote the sentence "player 1 wins after $2k$ stages". Then the formula

  $$\phi_1 := \exists x_1 \forall y_1 \ldots \exists x_k \forall y_k W$$

  denotes "player 1 has a winning strategy" and

  $$\phi_2 := \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \neg W$$

  denotes "player 2 has a winning strategy".
  But $\neg \phi_1 \equiv \phi_2$, i.e., $\phi_1 \vee \phi_2$ holds.