Social Networks Games

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Based on joint works with
Evangelos Markakis
and
Sunil Simon

Social Networks

- Facebook,
- Hyves,
- LinkedIn,
- Nasza Klasa,
- ...

But also ...

An area with links to

- sociology (spread of patterns of social behaviour)
- economics (effects of advertising, emergence of ‘bubbles’ in financial markets, ...),
- epidemiology (epidemics),
- computer science (complexity analysis),
- mathematics (graph theory).

Some Books

- C. P. Chamley. Rational herds: Economic models of social learning. Cambridge University Press, 2004.
- S. Goyal. Connections: An introduction to the economics of networks. Princeton University Press, 2007.
- F. Vega-Redondo. Complex Social Networks. Cambridge University Press, 2007.
- M. Jackson. Social and Economic Networks. Princeton University Press, Princeton, 2008.
- D. Easley and J. Kleinberg. Networks, Crowds, and Markets. Cambridge University Press, 2010.
- M. Newman. Networks: An Introduction. Oxford University Press, 2010.

Some Research Topics

- Spread of a disease.
- Viral marketing.
- Possible impact of a product.

Our model

Social network ([Apt, Markakis '11, '14])

- Weighted directed graph: $G = (V, \to, w)$, where
  V: a finite set of agents,
  $w_{ij} \in (0, 1]$: weight of the edge $i \to j$.
- Products: A finite set of products $\mathcal{P}$.
- Product assignment: $P : V \to 2^{\mathcal{P}} \setminus \{\emptyset\}$;
  assigns to each agent a non-empty set of products.
- Threshold function: $\theta(i, t) \in (0, 1]$, for each agent i and product $t \in P(i)$.

- Neighbours of node i: $\{j \in V \mid j \to i\}$.
- Source nodes: Agents with no neighbours.

Diffusion

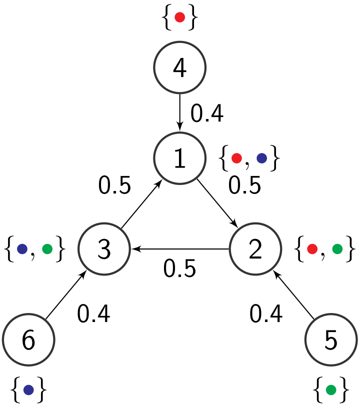
- Initially no node adopted any product.
- Source nodes can adopt any product from their product sets.
- A non-source node i can adopt some product t from its product set if

$$\sum_{j \in \mathcal{N}_i^t} w_{ji} > \theta(i, t),$$

  where

  $\mathcal{N}_i^t$: the set of neighbours of i who already adopted the product t.
- At each stage one or more nodes can adopt a product.
- The adopted choices are final.

Example



Threshold is 0.3 for all the players.

- $\mathcal{P} = \{$●, ●, ●$\}$

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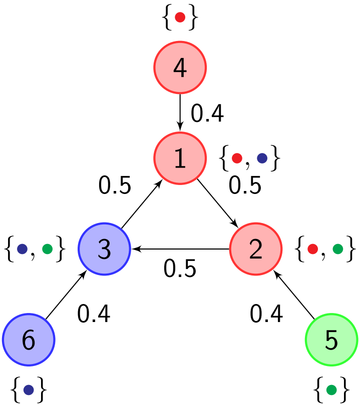

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This diffusion process can take between 3 and 6 steps.

Some Research Questions

Given an initial network and a product *top*.

MAX-ADOPTION: What is the maximum number of nodes that adopted *top* in a final network.

MIN-ADOPTION: What is the minimum number of nodes that adopted *top* in a final network.

Results

Theorem

- MAX-ADOPTION can be solved in $O(n^2)$ time.
- MIN-ADOPTION for 2 products can be solved in $O(n^2)$ time.
- For at least 3 products it is NP-hard to approximate MIN-ADOPTION with an approximation ratio better than $\Omega(n)$.

The associated strategic game

Interaction between agents: Each agent i can adopt a product from the set $P(i)$ or choose not to adopt any product (t_0).

Social network games

- Players: Agents in the network.
- Strategies: Set of strategies for player i is $P(i) \cup \{t_0\}$.
- Payoff: Fix $c > 0$.
  Given a joint strategy s and an agent i,

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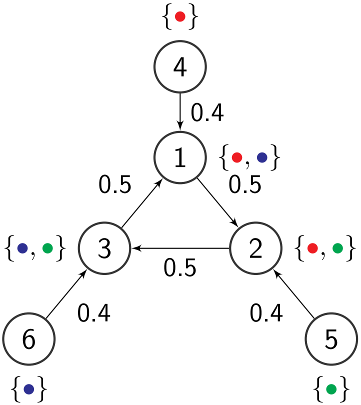
  - if $i \notin source(\mathcal{S})$, $p_i(s) =$
    $\begin{cases} 0 & \text{if } s_i = t_0 \\ \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i,t) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}$

$\mathcal{N}_i^t(s)$: the set of neighbours of i who adopted in s the product t.

Example



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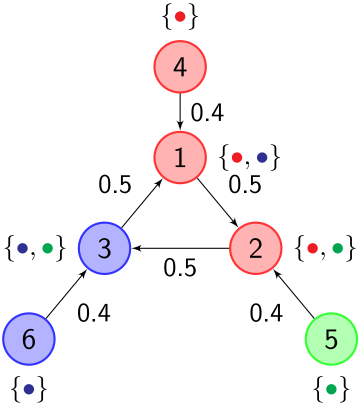

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Payoff:

- $p_4(s) = p_5(s) = p_6(s) = c$

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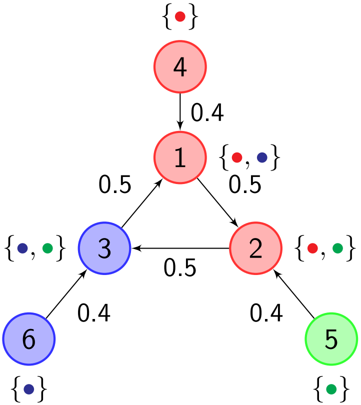

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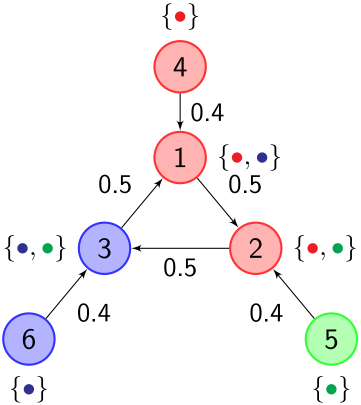

Payoff:

- $p_4(s) = p_5(s) = p_6(s) = c$
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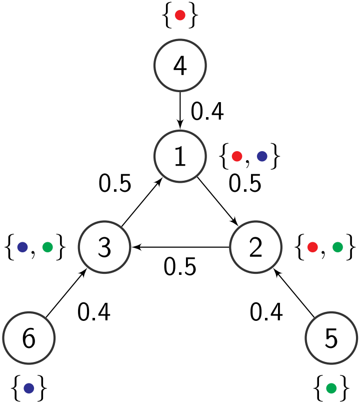
Social network games

Properties

- Graphical game: The payoff for each player depends only on the choices made by his neighbours.
- Join the crowd property: The payoff of each player weakly increases if more players choose the same strategy.

Does Nash equilibrium always exist?



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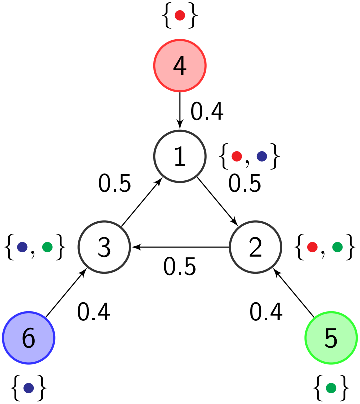

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Observation: No player has the incentive to choose t_0.

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- Each player on the cycle can ensure a payoff of at least 0.1.

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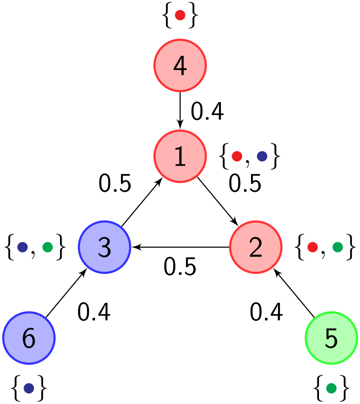

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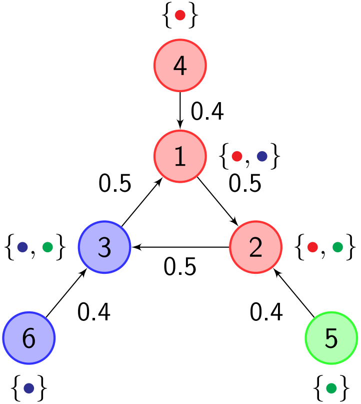

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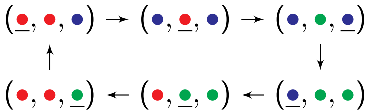
Best response dynamics



Observation: No player has the incentive to choose t_0.

- Source nodes can ensure a payoff of $c > 0$.
- Each player on the cycle can ensure a payoff of at least 0.1.

Reason: Players keep switching between the products.

Nash equilibrium

Question: Given a social network S, what is the complexity of deciding whether $G(S)$ has a Nash equilibrium?

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The PARTITION problem

Input: n positive rational numbers $(a_1, \ldots, a_n)$ such that $\sum_i a_i = 1$.

Question: Is there a set $S \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2}.$$

Hardness

Reduction: Given an instance of the PARTITION problem $P = (a_1, \ldots, a_n)$, construct a network $\mathcal{S}(P)$ such that there is a solution to P iff there is a Nash equilibrium in $\mathcal{S}(P)$.

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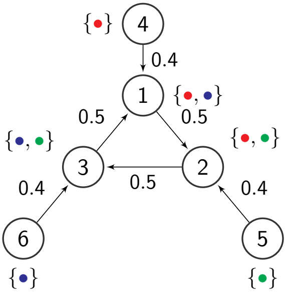

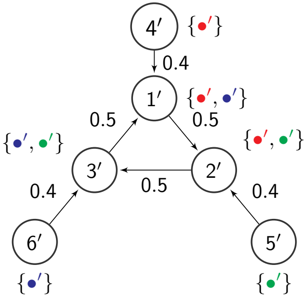

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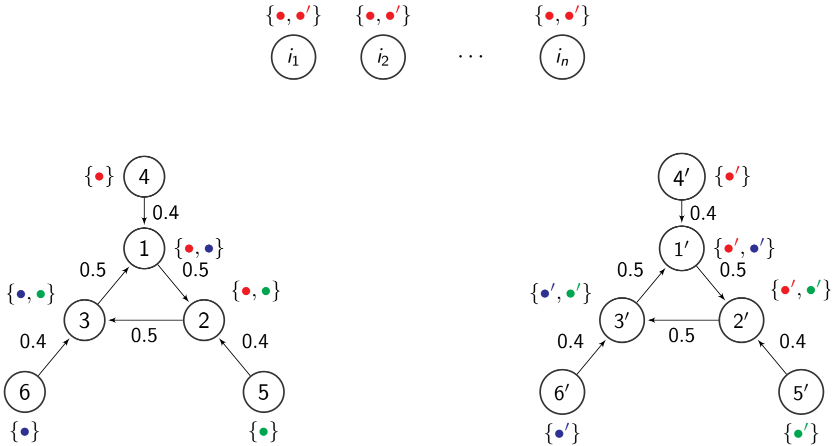

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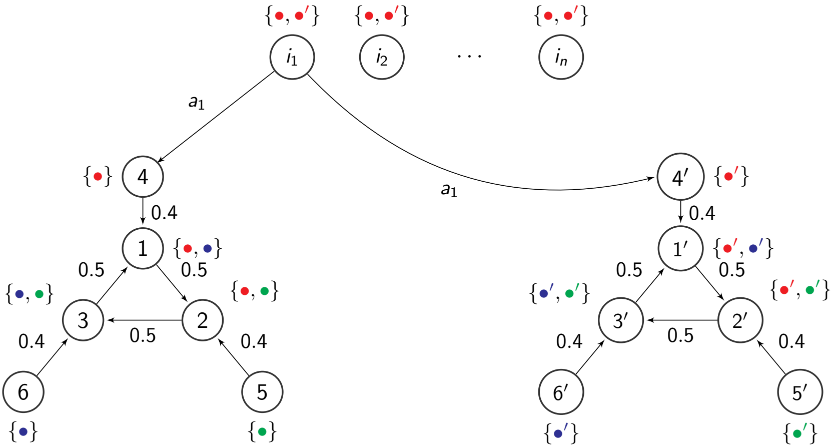

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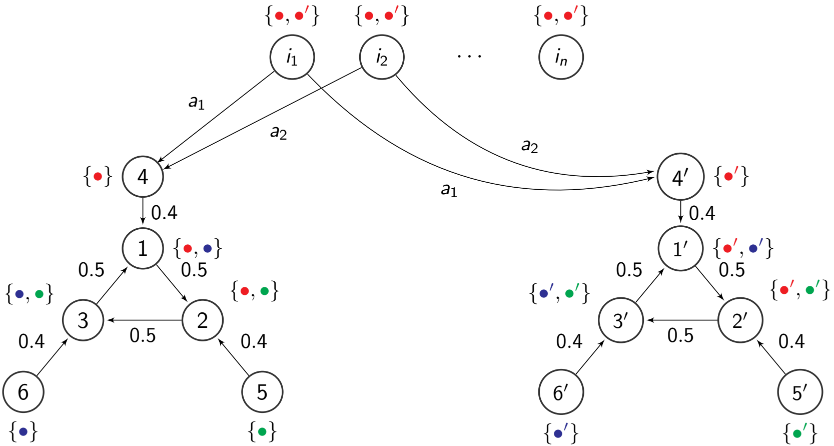

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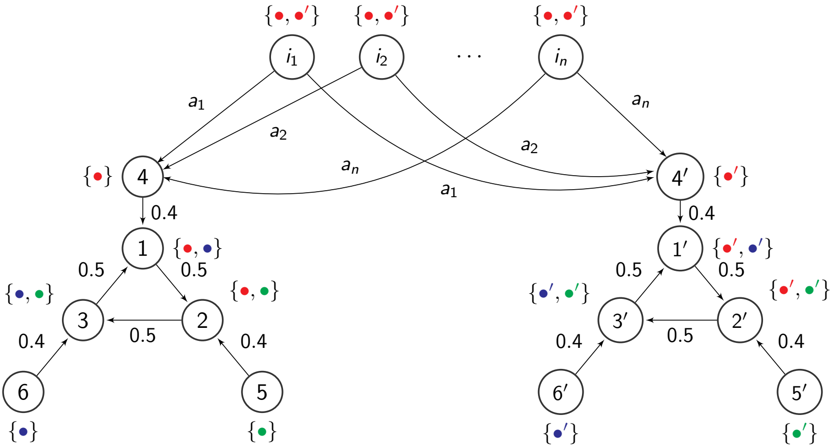
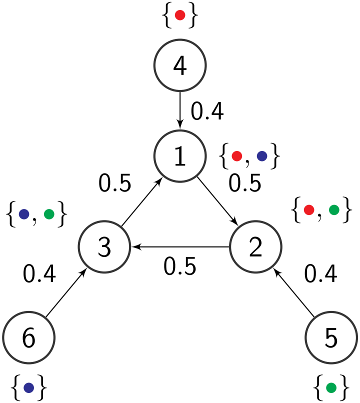


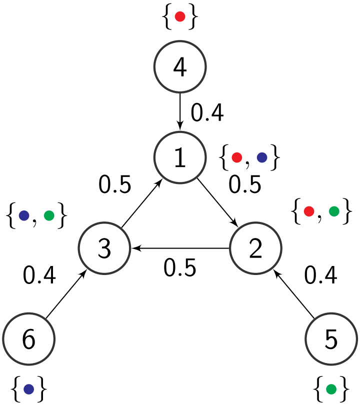
Nash equilibrium

Recall the network with no Nash equilibrium:



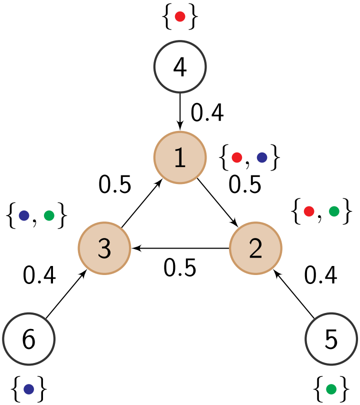
Theorem. If there are at most two products, then a Nash equilibrium always exists and can be computed in polynomial time.

Nash equilibrium



Properties of the underlying graph:

Nash equilibrium



Properties of the underlying graph:

- Contains a cycle.

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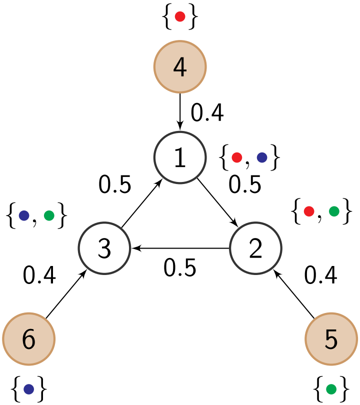
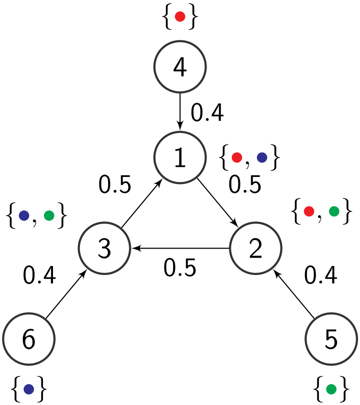


Properties of the underlying graph:

- Contains a cycle.
- Contains source nodes.

Nash equilibrium



Properties of the underlying graph:

- Contains a cycle.
- Contains source nodes.

Question: Does Nash equilibrium always exist in social networks when the underlying graph

- is acyclic?
- has no source nodes?

Non-trivial Nash equilibria

- A Nash equilibrium s is non-trivial if there is at least one player i such that $s_i \neq t_0$.
- Theorem. In a DAG, a non-trivial Nash equilibrium always exists.
- Theorem. Assume the graph has no source nodes. There is an algorithm with a running time $\mathcal{O}(|\mathcal{P}| \cdot n^3)$ that determines whether a non-trivial Nash equilibrium exists.

Finite Improvement Property

Fix a game.

- Profitable deviation: a pair (s, s') such that $s' = (s'_i, s_{-i})$ for some s'_i and $p_i(s') > p_i(s)$.
- Improvement path: a maximal sequence of profitable deviations.
- A game has the FIP if all improvement paths are finite.

Second Look at Diffusion

Note: Diffusion can be seen as a special case of an improvement path.

Comments

- Initial joint strategy: $(t_0, \ldots, t_0)$.
- Each player can change strategy only once.
- As a result diffusion can take only finitely many steps.
- The changes can take place simultaneously.

Summary of results

	arbitrary graphs	DAG	simple cycle	no source nodes
NE	NP-complete	always exists	always exists	always exists
Non-trivial NE	NP-complete	always exists	$\mathcal{O}(\|\mathcal{P}\| \cdot n)$	$\mathcal{O}(\|\mathcal{P}\| \cdot n^3)$
Determined NE	NP-complete	NP-complete	$\mathcal{O}(\|\mathcal{P}\| \cdot n)$	NP-complete

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FIP	co-NP-hard	yes	?	co-NP-hard
FBRP	co-NP-hard	yes	$\mathcal{O}(\lvert\mathcal{P}\rvert \cdot n)$	co-NP-hard
Weakly acyclic	co-NP-hard	yes	yes	co-NP-hard

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Weakly acyclic	co-NP-hard	yes	yes	co-NP-hard

FBRP: all improvement paths, in which only best responses are used, are finite.

Weakly acyclic: from every joint strategy there is a finite improvement path that starts at it.

Paradox of Choice (B. Schwartz, 2005)

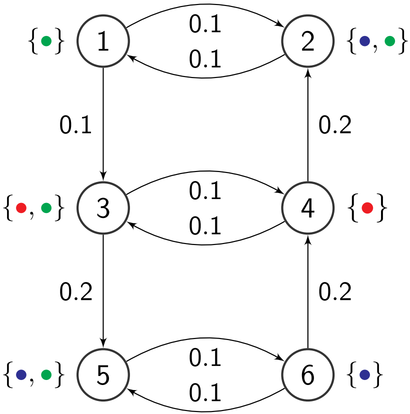
[*Gut Feelings*, G. Gigerenzer, 2008]

The more options one has, the more possibilities for experiencing conflict arise, and the more difficult it becomes to compare the options. There is a point where more options, products, and choices hurt both seller and consumer.

Paradox 1

Adding a product to a social network can trigger a sequence of changes that will lead the agents from one Nash equilibrium to a new one that is worse for everybody.

Example



- Cost θ is constant, $0 < \theta < 0.1$.

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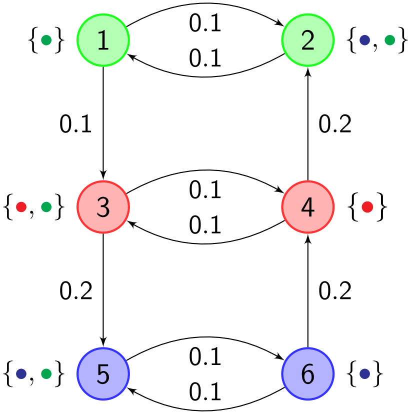

- Cost θ is constant, $0 < \theta < 0.1$.
- This is a Nash equilibrium. The payoff to each player is $0.1 - \theta > 0$.

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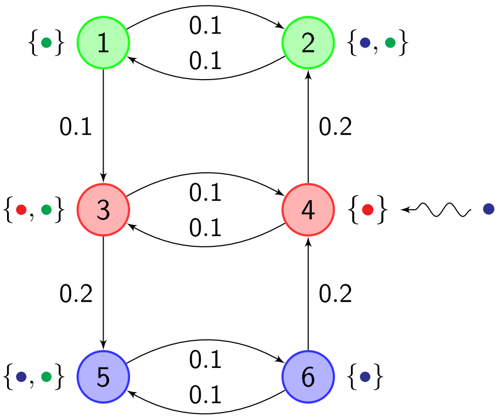

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- This is not a Nash equilibrium.

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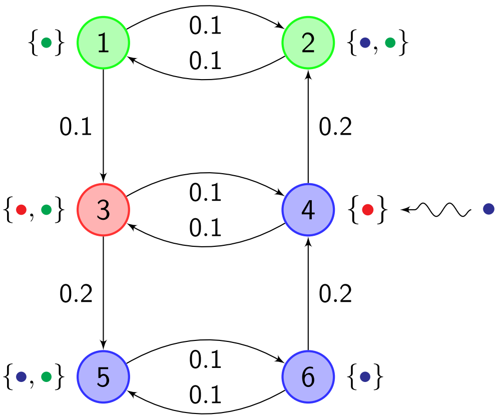

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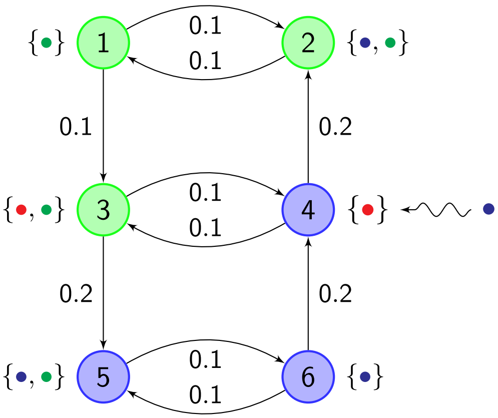

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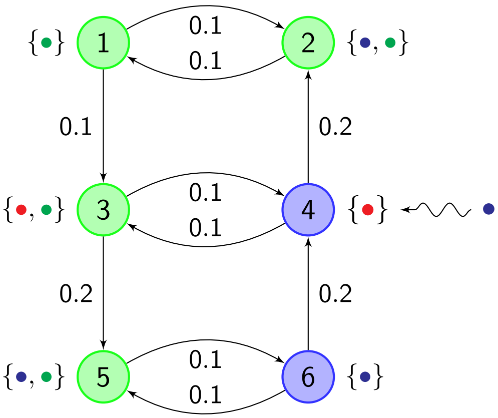

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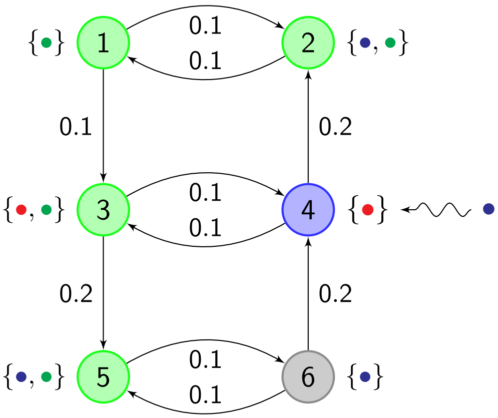

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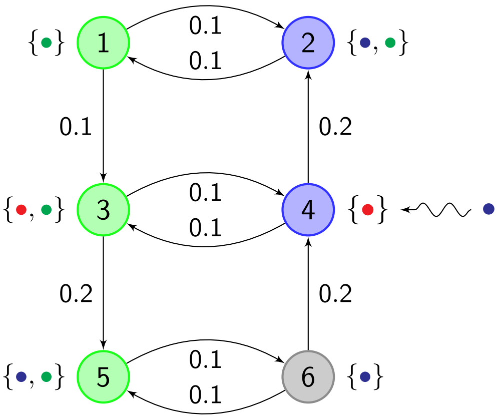

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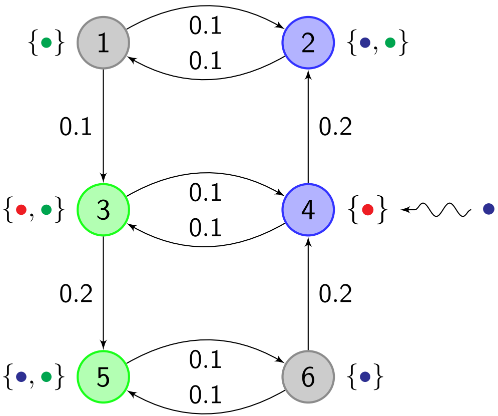

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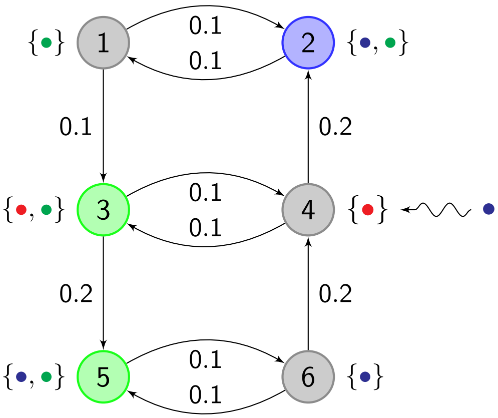

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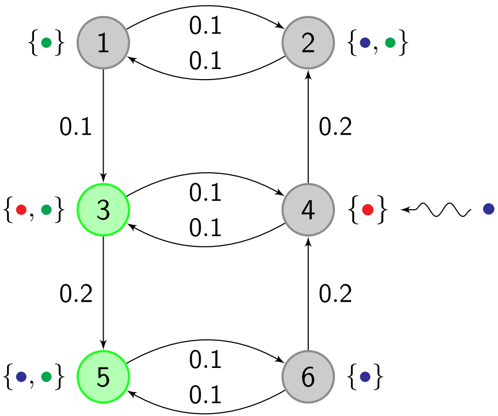

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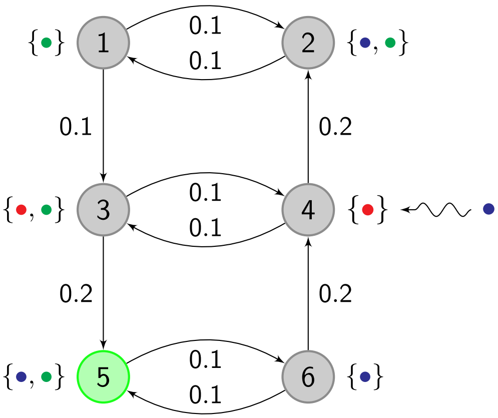

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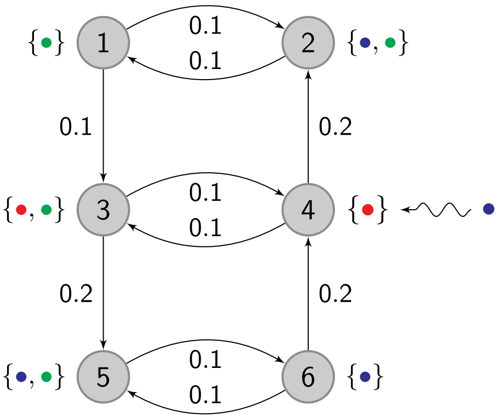

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- This is a Nash equilibrium. The payoff to each player is 0.

Paradox 2

Removing a product from a social network can result in a sequence of changes that will lead the agents from one Nash equilibrium to a new one that is better for everybody.

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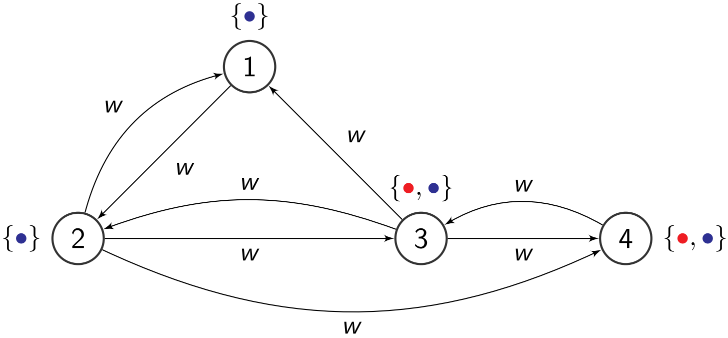
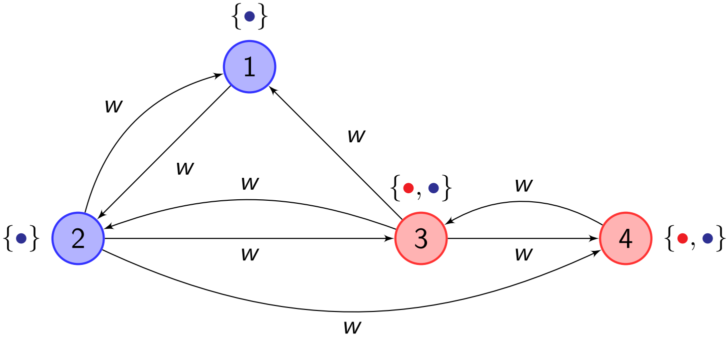


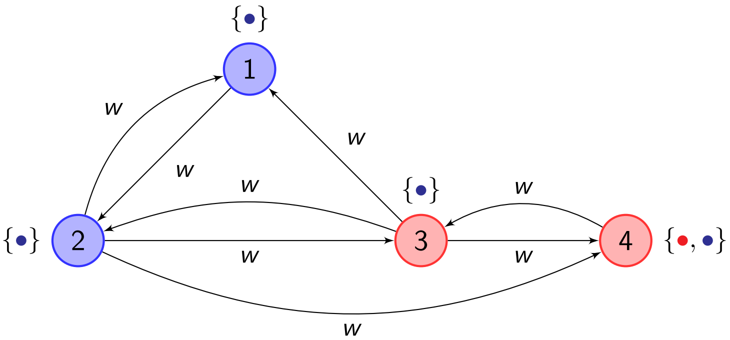
- Cost θ is product independent.
- The weight of each edge is w, where $w > \theta$.
- Note Each node has two incoming edges.

Example



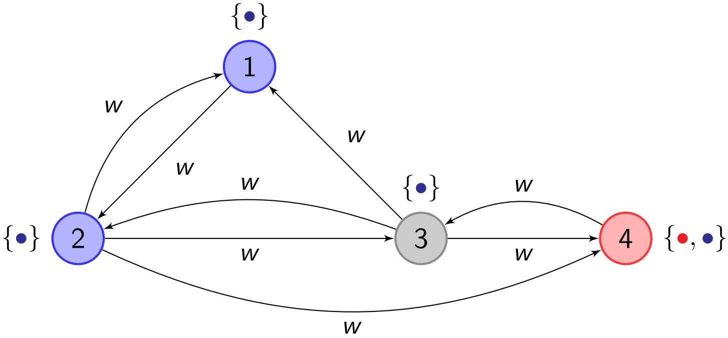
- Cost θ is product independent.
- The weight of each edge is w, where $w > \theta$.
- This is a Nash equilibrium. The payoff to each player is $w - \theta$.

Example



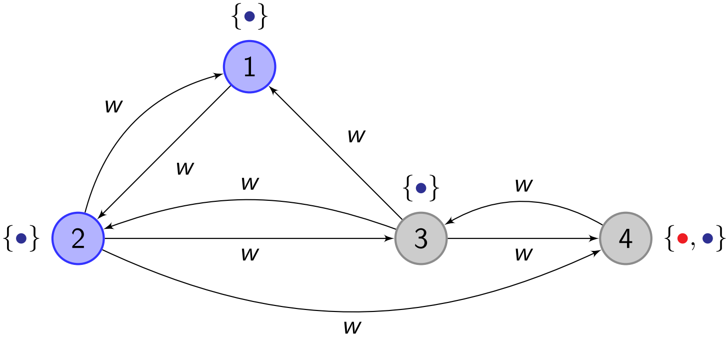
- Cost θ is product independent.
- The weight of each edge is w, where $w > \theta$.
- This is not a legal joint strategy.

Example



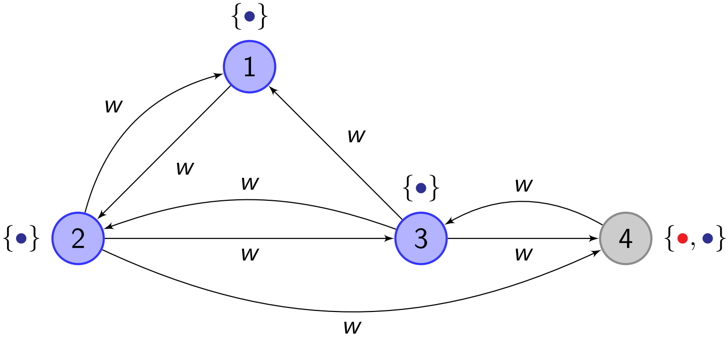
- Cost θ is product independent.
- The weight of each edge is w, where $w > \theta$.
- This is not a Nash equilibrium.

Example



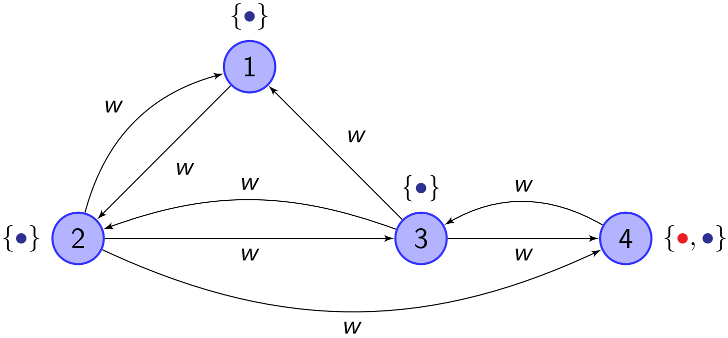
- Cost θ is product independent.
- The weight of each edge is w, where $w > \theta$.
- This is not a Nash equilibrium.

Example



- Cost θ is product independent.
- The weight of each edge is w, where $w > \theta$.
- This is not a Nash equilibrium.

Example



- Cost θ is product independent.
- The weight of each edge is w, where $w > \theta$.
- This is a Nash equilibrium. The payoff to each player is $2w - \theta$.

Open problem

- Does a social network exist that exhibits paradox 1 for every triggered sequence of changes?
- Alternative approach:
  Obligatory product selection (no t_0).
  In this setup the above problem has an affirmative answer.

References

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A follow up

- Drop the thresholds, weights and t_0. Consider undirected graphs.
- The resulting games: coordination games on graphs.
- payoff to a player = number of neighbours who chose his product.
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- ... and a number of other publications by M. Rahn, G. Schaefer, S. Simon and D. Wojtczak.