Selfishness Level of Strategic Games

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Altruistic Games

- Given $G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$ and $\alpha \geq 0$.
- Social welfare of s:

$$SW(s) := \sum_{j=1}^{n} p_j(s).$$

- $G(\alpha) := (N, \{S_i\}_{i \in N}, \{r_i\}_{i \in N})$, where

$$r_i(s) := p_i(s) + \alpha SW(s).$$

- When $\alpha > 0$ the payoff of each player in $G(\alpha)$ depends on the social welfare of the players.
- $G(\alpha)$ is an altruistic version of G.

Selfishness Level (1)

- G is α-selfish if a Nash equilibrium of $G(\alpha)$ is a social optimum of $G(\alpha)$.
- Selfishness level of G:

$$\inf\{\alpha \in \mathbb{R}_+ \mid G \text{ is } \alpha\text{-selfish}\}.$$

  Recall $\inf(\emptyset) = \infty$.
- Selfishness level of G is α^+ iff the selfishness level of G is $\alpha \in \mathbb{R}_+$ but G is *not* α-selfish.

Selfishness Level (2)

Intuition

Selfishness level quantifies the minimal share of social welfare needed to induce the players to choose a social optimum.

Three Examples (1)

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Three Examples (2)

Prisoner's Dilemma: selfishness level is 1.

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

	C	D
C	$6, 6$	$3, 6$
D	$6, 3$	$3, 3$

The Battle of the Sexes: selfishness level is 0.

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies: selfishness level is ∞.

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Another Example

Game with a bad Nash equilibrium

	H	T	E
H	$1, -1$	$-1, \quad 1$	$-1, -1$
T	$-1, \quad 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

- The unique Nash equilibrium is (E, E).
- The selfishness level of this game is ∞.

Yet Another Example

Game with no Nash equilibrium

Consider G on the left and $G(1)$ on the right.

	C	D
C	$2, 2$	$2, 0$
D	$3, 0$	$1, 1$

	C	D
C	$6, 6$	$4, 2$
D	$6, 3$	$3, 3$

- G has no Nash equilibrium, while in $G(1)$ the social optimum, (C, C), is also a Nash equilibrium.
- For $\alpha < 1$, (C, C) is also a social optimum of $G(\alpha)$ but not a Nash equilibrium.
- So the selfishness level of the game G is 1.

Invariance of Selfishness Level

Lemma Consider a game G and $\alpha \geq 0$.

- For every a, G is α-selfish iff $G + a$ is α-selfish,
- For every $a > 0$, G is α-selfish iff aG is α-selfish.

Conclusion Selfishness level is invariant under positive linear transformations of the payoff functions.

Selfishness Level vs Price of Stability (1)

- Recall
  Price of stability $= SW(s)/\, SW(s')$,
  where s is a social optimum and s' a Nash equilibrium with the highest social welfare.
- Note
  Selfishness level of a finite game is 0 iff price of stability is 1.

Selfishness Level vs Price of Stability (2)

Theorem For every finite $\alpha > 0$ and $\beta > 1$ there is a finite game with selfishness level α and price of stability β.

Proof Consider G:

	C	D
C	$1, 1$	$0, \frac{2\alpha+1}{\alpha+1}$
D	$\frac{2\alpha+1}{\alpha+1}, 0$	$\frac{1}{\beta}, \frac{1}{\beta}$

In each $G(\gamma)$ with $\gamma \geq 0$, (C, C) is the unique social optimum.
Consider $G(\gamma)$ and stipulate that (C, C) is its Nash equilibrium. This leads to

$$1 + 2\gamma \geq (\gamma + 1)\frac{2\alpha + 1}{\alpha + 1}.$$

This is equivalent to $\gamma \geq \alpha$. So the selfishness level is α.
The price of stability is β.

Selfishness Level can be α^+

Theorem There exists a game that is 0^+-selfish (so α-selfish for every $\alpha > 0$, but is not 0-selfish).

Proof idea

Plug the above games for each $\alpha > 0$ and fixed $\beta > 1$ in:

...	...	0	0	0	0
...	...	0	0	0	0
0	0	...	...	0	0
0	0	...	...	0	0
0	0	0	0	...	...
0	0	0	0	...	...

Alternative Definitions (1)

A: For every $\alpha \geq 0$, $G(\alpha) := (N, \{S_i\}_{i \in N}, \{r_i^{\alpha}\}_{i \in N})$ with

$$r_i^{\alpha}(s) = p_i(s) + \alpha SW(s) \quad \forall i \in N.$$

B: For every $\beta \in [0,1]$, $G(\beta) := (N, \{S_i\}_{i \in N}, \{r_i^{\beta}\}_{i \in N})$ with

$$r_i^{\beta}(s) = (1-\beta)p_i(s) + \frac{\beta}{n} SW(s) \quad \forall i \in N.$$

C: For every $\gamma \in [0,1]$, $G(\gamma) := (N, \{S_i\}_{i \in N}, \{r_i^{\gamma}\}_{i \in N})$ with

$$r_i^{\gamma}(s) = (1-\gamma)p_i(s) + \gamma SW(s) \quad \forall i \in N.$$

D: For every $\delta \in [0,1]$, $G(\delta) := (N, \{S_i\}_{i \in N}, \{r_i^{\delta}\}_{i \in N})$ with

$$r_i^{\gamma}(s) = (1-\delta)p_i(s) + \delta(SW(s) - p_i(s)) \quad \forall i \in N.$$

Alternative Definitions (2)

Theorem

Consider $G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$ and its altruistic versions defined according to models A, B, C and D.

(i) G is α-selfish with $\alpha \in \mathbb{R}_+$ iff G is β-selfish with $\beta = \frac{\alpha n}{1+\alpha n} \in [0, 1]$.

(ii) G is α-selfish with $\alpha \in \mathbb{R}_+$ iff G is γ-selfish with $\gamma = \frac{\alpha}{1+\alpha} \in [0, 1]$.

(iii) G is α-selfish with $\alpha \in \mathbb{R}_+$ iff G is δ-selfish with $\delta = \frac{\alpha}{1+2\alpha} \in [0, \frac{1}{2}]$.

Stable Social Optima

- Social optimum s stable if no player is better off by unilaterally deviating to another social optimum.
- That is, s is stable if for all $i \in N$ and $s'_i \in S_i$

$$\text{if } (s'_i, s_{-i}) \text{ is a social optimum, then } p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- Notes
  - If s is a unique social optimum, then it is stable.
  - Stable social optima don't need to exist:
    take the Matching Pennies game.

Characterization Result

Player i's appeal factor of s'_i given the social optimum s:

$$AF_i(s'_i, s) := \frac{p_i(s'_i, s_{-i}) - p_i(s_i, s_{-i})}{SW(s_i, s_{-i}) - SW(s'_i, s_{-i})}.$$

Theorem

- The selfishness level of G is finite iff a stable social optimum s exists for which $\alpha(s) := \max_{i \in N,\, s'_i \in U_i(s)} AF_i(s'_i, s)$
  is finite, where
  $U_i(s) := \{s'_i \in S_i \mid p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})\}$.
- If the selfishness level of G is finite, then it equals $\min_{s \in SSO} \alpha(s)$, where SSO is the set of stable social optima.

Some Observations

- If G is finite, then its selfishness level is finite iff it has a stable social optimum.
- Selfishness level can be unbounded.

  Theorem For each $f : \mathbb{N} \rightarrow \mathbb{R}_+$ there exists a class of games G_n for n players, such that the selfishness level of G_n is $f(n)$.

Prisoner's dilemma for n players

- $S_i = \{0, 1\}$,
- b: the benefit of cooperation,
  c: the cost of cooperation,
- $b > c$,
- $p_i(s) := -cs_i + b\sum_{j \neq i} s_j$.

Proposition Selfishness level is $\frac{c}{b(n-1)-c}$.

Notes

- When n increases, a smaller share of the social welfare is needed to resolve the underlying conflict.
- The same for the value b of the benefit.
- So the 'acuteness' of the dilemma diminishes with the number of players and also when the value of the benefit grows.

Public Goods Game

- n players,
- $b \in \mathbb{R}_+$: fixed budget,
- $c > 1$: a multiplier,
- $S_i = [0, b]$,
- $p_i(s) := b - s_i + \frac{c}{n} \sum_{j \in N} s_j$.

Proposition Selfishness level is $\max\left\{0, \frac{1-\frac{c}{n}}{c-1}\right\}$.
Notes

- Free riding: contributing 0 (it is a dominant strategy).
- For fixed c temptation to free ride increases with n.
- For fixed n temptation to free ride decreases as c increases.

Potential Games

$$G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$$

is a generalized ordinal potential game if for some $P : S_1 \times \ldots \times S_n \to \mathbb{R}$ for all $i \in N$, $s_{-i} \in S_{-i}$ and $s_i, s'_i \in S_i$

$$p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}) \text{ implies } P(s_i, s_{-i}) > P(s'_i, s_{-i}).$$

Theorem Every finite generalized ordinal potential game has a finite selfishness level.

Proof Each social optimum with the largest potential is a stable social optimum.

Fair Cost Sharing Games (1)

Fair cost sharing game: $G = (N, E, \{S_i\}_{i \in N}, \{c_e\}_{e \in E})$, where

- E is the set of facilities,
- $S_i \subseteq 2^E$ is the set of facility subsets available to player i, i.e., each $s_i \subseteq E$,
- $c_e \in \mathbb{Q}_+$ is the cost of facility $e \in E$.
- Let $x_e(s)$ be the number of players using facility e in s.
- The cost of facility $e \in E$ is evenly shared. So $c_i(s) := \sum_{e \in s_i} \frac{c_e}{x_e(s)}$.
- Social cost: $SC(s) = \sum_{i=1}^{n} c_i(s)$.

Fair Cost Sharing Games (2)

Singleton cost sharing game: for each s_i, $|s_i| = 1$.

- $c_{\max} := \max_{e \in E} c_e$,
- $c_{\min} := \min_{e \in E} c_e$,
- $L := \max_{i \in N,\ s_i \in S_i} |s_i|$
  (maximum number of facilities a player can choose).

Proposition Selfishness level of

- a singleton cost sharing game is $\leq \max\{0, \frac{1}{2}\frac{c_{\max}}{c_{\min}} - 1\}$. The bound is tight.
- a fair cost sharing game with non-negative integer costs is $\leq \max\{0, \frac{1}{2} L c_{\max} - 1\}$. The bound is tight.

Congestion Games

Congestion game: $G = (N, E, \{S_i\}_{i \in N}, \{d_e\}_{e \in E})$,
where

- E is a finite set of facilities,
- $S_i \subseteq 2^E$ is the set of facility subsets available to player i,
- $d_e \in \mathbb{N}$ is the delay function for facility $e \in E$.
- Let $x_e(s)$ be the number of players using facility e in s.
- The goal of a player is to minimize his individual cost $c_i(s) := \sum_{e \in s_i} d_e(x_e(s))$.
- Social cost:
  $SC(s) = \sum_{i=1}^{n} c_i(s)$.

Linear Congestion Games

Linear congestion game: each delay function is of the form $d_e(x) = a_e x + b_e$, where $a_e, b_e \in \mathbb{R}_+$.

- $\Delta_{\max} := \max_{e \in E}(a_e + b_e)$,
- $\Delta_{\min} := \min_{e \in E}(a_e + b_e)$,
- $L := \max_{i \in N,\ s_i \in S_i} |s_i|$
  (maximum number of facilities a player can choose).

Proposition Selfishness level of

- a linear congestion game with non-negative integer coefficients is $\leq \max\{0, \frac{1}{2}(L\Delta_{\max} - \Delta_{\min} - 1)\}$. This bound is tight.

Cournot Competition (1)

- One infinitely divisible product (oil),
- n companies decide simultaneously how much to produce,
- price is decreasing in total output.

Each $S_i = \mathbb{R}_+$,

$$p_i(s) := s_i\left(a - b\sum_{j=1}^{n} s_j\right) - cs_i$$

for some a, b, c, where $a > c$ and $b > 0$.

The price of the product: $a - b\sum_{j=1}^{n} s_j$.

The production cost: cs_i.

Cournot Competition (2)

- $p_i(s) := s_i\left(a - b\sum_{j=1}^{n} s_j\right) - cs_i$
- Unique Nash equilibrium:
  s, with each $s_i = \frac{a-c}{b(n+1)}$.
  $SW(s) = \frac{(a-c)^2}{b} \cdot \frac{n}{(n+1)^2}$.
- Social optimum, when $\sum_{j=1}^{n} s_j = \frac{a-c}{2b}$.
  $SW(s) = \frac{(a-c)^2}{4b}$.
- Note Price of stability converges to ∞.

Proposition For each $n > 1$ the selfishness level is ∞.

Tragedy of the Commons (1)

- Contiguous common resource (e.g. shared bandwidth),
- Each $S_i = [0, 1]$,
- s_i: chosen fraction of the common resource
- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Intuition: the payoff degrades when the resource is overused.

Tragedy of the Commons (2)

- $p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^n s_j) & \text{if } \sum_{j=1}^n s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$
- Best Nash equilibrium:
  s, with each $s_i = \frac{1}{n+1}$.
  $SW(s) = \frac{n}{(n+1)^2}$.
- Social optimum, when $\sum_{j=1}^n s_j = \frac{1}{2}$.
  $SW(s) = \frac{1}{4}$.
- Note Price of stability converges to ∞.

Proposition For each $n > 1$ the selfishness level is ∞.

Bertrand Competition

- One product for sale.
- 2 companies simultaneously select their prices.
- The product is sold by the company that chose a lower price.

Each $S_i = [c, \frac{a}{b})$, where $c < \frac{a}{b}$.
(So $s_i - c \geq 0$ and $a - bs_i > 0$ for $s_i \in S_i$.)

$$p_i(s_i, s_{3-i}) := \begin{cases} (s_i - c)(a - bs_i) & \text{if } c < s_i < s_{3-i} \\ \frac{1}{2}(s_i - c)(a - bs_i) & \text{if } c < s_i = s_{3-i} \\ 0 & \text{otherwise.} \end{cases}$$

The demand for the product: $a - bs_i$.

The marginal production cost: c.

Proposition The selfishness level is ∞.

Mixed Nash Equilibria

- Suppose now:
  G is α-selfish if a mixed Nash equilibrium of $G(\alpha)$ is a social optimum.
- Define selfishness level in mixed strategies as before.
- Then the selfishness level of the Matching Pennies game is 0.
- A finite selfishness level can decrease when we use mixed Nash equilibria.

Mixed Nash Equilibria, ctd

Example

	H	T	C	D
H	$3, 1$	$1, 3$	$0, 0$	$0, 0$
T	$1, 3$	$3, 1$	$0, 0$	$0, 0$
C	$0, 0$	$0, 0$	$2, 2$	$0, 3$
D	$0, 0$	$0, 0$	$3, 0$	$1, 1$

- This game has a unique stable social optimum, (C, C), and a unique pure Nash equilibrium, (D, D).
- Its selfishness level is 1.
- $(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T)$ is both a mixed Nash equilibrium and a social optimum in mixed strategies.
- So the selfishness level in mixed strategies is 0.

Some Quotations

- The intelligent way to be selfish is to work for the welfare of others.

  Microeconomics: Behavior, Institutions, and Evolution, S. Bowles '04.

- An excellent way to promote cooperation in a society is to teach people to care about the welfare of others.

  The Evolution of Cooperation, R. Axelrod, '84.

Reference

- K.R. Apt, Guido Schäfer, *Selfishness Level of Strategic Games*, Journal of AI Research (JAIR) 49, pp. 207-240 (2014).