Solutions to the problem set for

A Crash Course in Strategic Games

1. Show that in the location game from Example 2.16 no weakly or strongly dominant strategy exists.

Solution

Every weakly or strictly dominant strategy is also a dominant strategy. So it suffices to prove that in this game no dominant strategy exists.

Each dominant strategy is a best response to every joint strategy of the opponents. The game under consideration has two players and is symmetric. So it suffices to exhibit a strategy of player 2 for which no best response of player 1 exists. Any strategy $s_2 \neq 50$ will do, for example $s_2 = 60$. Indeed, if $s_1 = 60$, then 50 is a better response and if $s_1 \neq 60$, then $\frac{s_1+60}{2}$ is a better response.

Another way to prove it is by showing that no strategy of player 1 is a best response to *every* strategy of player 2. As noticed in Example 2.16 the only strategy of player 1 that is a best response to *some* strategy of player 2 is 50. But 50 is not a best response to 60, since 51 is a better response to 60. (In fact 50 is a best response only to 50.)

2. Prove that

$$P(s) := s_1 s_2 \ldots s_n (a - b \sum_{j=1}^{n} s_j - c)$$

is an ordinal potential for the Cournot competition game introduced in Example 1.7 and analyzed in Example 3.4.

Solution

Recall that for $i \in \{1, \ldots, n\}$ we have

$$p_i(s) = s_i(a - b \sum_{j=1}^{n} s_j) - cs_i = s_i(a - b \sum_{j=1}^{n} s_j - c).$$

So $i \in \{1, \ldots, n\}$ we have

$$P(s) = s_1 s_2 \ldots s_{i-1} s_{i+1} \ldots s_n p_i(s).$$

Hence

$$\forall i \in \{1, \ldots, n\} \ \forall s_{-i} \in S_{-i} \ \forall s_i, s'_i \in S_i$$
$$s_1 s_2 \ldots s_{i-1} s_{i+1} \ldots s_n (p_i(s_i, s_{-i}) - p_i(s'_i, s_{-i})) = P(s_i, s_{-i}) - P(s'_i, s_{-i}).$$

The conclusion now follows since for $i \in \{1, \ldots, n\}$ we have $s_1 s_2 \ldots s_{i-1} s_{i+1} \ldots s_n > 0$ as by assumption all considered strategies are from $\mathbb{R}_+$.

3. Consider the congestion game represented as a network in Figure 1. The delay functions on the arcs/facilities are either constant (4 or 5) or equal to the number of players who choose the arc (denoted by T).

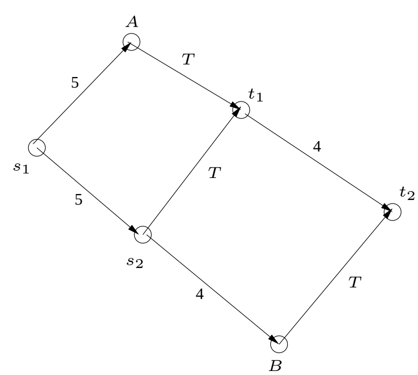

Figure 1: A network

There are 6 players who need to choose a path from s_1 to t_1 and 6 players who need to choose a path from s_2 to t_2. So each of the drivers in the first set has two strategies, corresponding respectively to the paths $s_1 \to A \to t_1$ and $s_1 \to s_2 \to t_1$, while each of the drivers in the second set has two strategies, corresponding respectively to the paths $s_2 \to t_1 \to t_2$ and $s_2 \to B \to t_2$.

Consider a joint strategy. Denote by

- T_1 the number of players who chose the path $s_1 \to A \to t_1$,
- T_2 the number of players who chose the path $s_1 \to s_2 \to t_1$,
- T_3 the number of players who chose the path $s_2 \to t_1 \to t_2$,
- T_4 the number of players who chose the path $s_2 \to B \to t_2$.

By assumption we have

$$T_1 + T_2 = 6,\ T_3 + T_4 = 6.$$

(i) Write the conditions on T_1, T_2, T_3, T_4 that determine that a joint strategy is a Nash equilibrium.

(ii) Write the expression in T_1, T_2, T_3, T_4 that defines the social cost of a joint strategy.

(iii) Compute the price of anarchy and the price of stability for this game.

Solution

Note that T_2+T_3 is then the number of drivers who took the road segment $s_2 \to t_1$. Consequently, the considered strategy is a Nash equilibrium iff the following constraints are satisfied for the drivers who need to choose a road from s_1 to t_1:

- for changing the road $s_1 \to s_2 \to t_1$ to $s_1 \to A \to t_1$:

  $T_2 > 0 \to 5 + T_1 + 1 \geq 5 + T_2 + T_3$,

- for changing the road $s_1 \to A \to t_1$ to $s_1 \to s_2 \to t_1$:

  $T_1 > 0 \to 5 + T_2 + T_3 + 1 \geq 5 + T_1$,

and the following constraints are satisfied for the drivers who need to choose a road from s_2 to t_2:

- for changing the road $s_2 \to B \to t_2$ to $s_2 \to t_1 \to t_2$:

  $T_4 > 0 \to T_2 + T_3 + 4 + 1 \geq 4 + T_4$,

- for changing the road $s_2 \to t_1 \to t_2$ to $s_2 \to B \to t_2$:

  $T_3 > 0 \to 4 + T_4 + 1 \geq T_2 + T_3 + 4$.

Further, the social cost of the considered joint strategy equals

$$(5 + T_1)T_1 + (5 + T_2 + T_3)T_2 + (T_2 + T_3 + 4)T_3 + (4 + T_4)T_4.$$

One can check (we did it using the programming language ECL^{i}PS^e) that there are three ways of satisfying the above constraints:

- $T_1 = 3, T_2 = 3, T_3 = 1, T_4 = 5$, with the social cost 104,
- $T_1 = 4, T_2 = 2, T_3 = 2, T_4 = 4$, with the social cost 102,

- $T_1 = 5, T_2 = 1, T_3 = 3, T_4 = 3$, with the social cost 104.

The second Nash equilibrium is also a social optimum. Consequently, the price of anarchy of this game equals $\frac{104}{102}$, while the price of stability equals 1.