Nash and Von Neumann Theorems

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Overview

- Mixed strategies.
- Mixed extension of a finite game.
- Nash Theorem.

Example: Battle of Sexes

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

- Suppose player 1 (man) chooses F with the probability $\frac{1}{2}$ and B with the probability $\frac{1}{2}$.
  We write it as $\frac{1}{2}F + \frac{1}{2}B$.
- Similarly, suppose player 2 (woman) chooses F with the probability $\frac{1}{4}$ and B with the probability $\frac{3}{4}$.
  We write it as $\frac{1}{4}F + \frac{3}{4}B$.
- These are examples of mixed strategies.
- We call the previous strategies pure strategies.

Example: Battle of Sexes (2)

Suppose:
player 1 (man) chooses $\frac{1}{2}F + \frac{1}{2}B$,
player 2 (woman) chooses $\frac{1}{4}F + \frac{3}{4}B$.

	F	B
F	$\frac{1}{8}$	$\frac{3}{8}$
B	$\frac{1}{8}$	$\frac{3}{8}$

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

The payoffs:

$$p_1(m) = \frac{1}{8}2 + \frac{3}{8}0 + \frac{1}{8}0 + \frac{3}{8}1 = \frac{5}{8}.$$

$$p_2(m) = \frac{1}{8}1 + \frac{3}{8}0 + \frac{1}{8}0 + \frac{3}{8}2 = \frac{7}{8}.$$

Mixed Extension of a Finite Game

- Probability distribution over a finite non-empty set A:

$$\pi : A \to [0, 1]$$

  such that $\sum_{a \in A} \pi(a) = 1$.
- Notation: ΔA.

Fix a finite strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.

- Mixed strategy of player i in G: $m_i \in \Delta S_i$.
- Joint mixed strategy: $m = (m_1, \ldots, m_n)$.

Mixed Extension of a Finite Game (2)

- Mixed extension of G:

$$(\Delta S_1, \ldots, \Delta S_n, p_1, \ldots, p_n),$$

  where

$$m(s) := m_1(s_1) \cdot \ldots \cdot m_n(s_n)$$

  and

$$p_i(m) := \sum_{s \in S} m(s) \cdot p_i(s).$$

- Theorem (Nash '50)
  Every mixed extension of a finite strategic game has a Nash equilibrium.

2 Examples

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

- $(\frac{1}{2} \cdot H + \frac{1}{2} \cdot T, \frac{1}{2} \cdot H + \frac{1}{2} \cdot T)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: 0.

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

- $(2/3 \cdot F + 1/3 \cdot B, 1/3 \cdot F + 2/3 \cdot B)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: $2/3$.

Lemma

Consider a finite $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. The following are equivalent:

- m is a Nash equilibrium in mixed strategies,
  i.e.,
  $$p_i(m) \geq p_i(m'_i, m_{-i})$$
  for all $i \in \{1, \ldots, n\}$ and all $m'_i \in \Delta S_i$,
- for all $i \in \{1, \ldots, n\}$ and all $s_i \in S_i$
  $$p_i(m) \geq p_i(s_i, m_{-i}).$$

This equivalence implies that each Nash equilibrium of the initial game is a pure Nash equilibrium of the mixed extension.

Kakutani's Fixed Point Theorem

Theorem (Kakutani '41)
Suppose A is a compact and convex subset of $\mathbb{R}^n$ and

$$\Phi : A \to \mathcal{P}(A)$$

is such that

- $\Phi(x)$ is non-empty and convex for all $x \in A$,
- for all sequences (x_i, y_i) converging to (x, y)

  $$y_i \in \Phi(x_i) \text{ for all } i \geq 0,$$

  implies that

  $$y \in \Phi(x).$$

Then $x^* \in A$ exists such that $x^* \in \Phi(x^*)$.

Proof of Nash Theorem

Fix $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Define

$$best_i : \Pi_{j \neq i} \Delta S_j \to \mathcal{P}(\Delta S_i)$$

by

$$best_i(m_{-i}) := \{m'_i \in \Delta S_i \mid p_i(m'_i, m_{-i}) \text{ attains the maximum}\}.$$

Then define

$$best : \Delta S_1 \times \cdots \times \Delta S_n \to \mathcal{P}(\Delta S_1 \times \cdots \times \Delta S_n)$$

by

$$best(m) := best_1(m_{-1}) \times \cdots \times best_n(m_{-n}).$$

Note m is a Nash equilibrium iff $m \in best(m)$.

Proof of Nash Theorem, ctd

- $best_i(m_{-i}) := \{m'_i \in \Delta S_i \mid p_i(m'_i, m_{-i})$ attains the maximum$\}$. and

$$best(m) := best_1(m_{-1}) \times \cdots \times best_n(m_{-n}).$$

- $best(\cdot)$ satisfies the conditions of Kakutani's Theorem.
- $best(m)$ is non-empty.

Extreme Value Theorem
Suppose that A is a non-empty compact subset of $\mathbb{R}^n$ and

$$f : A \to \mathbb{R}$$

is a continuous function. Then f attains a minimum and a maximum.

Overview

- Security strategies.
- Strictly competitive games.
- Zero-sum games.

Security Strategies

Fix a (possibly infinite) game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
Assume the considered minima and maxima always exist.

- Consider

$$f_i(s_i) := \min_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}).$$

- We call any strategy s_i^* such that

$$f_i(s_i^*) = \max_{s_i \in S_i} f_i(s_i)$$

  a security strategy for player i.

- Denote

$$maxmin_i := \max_{s_i \in S_i} f_i(s_i) = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}).$$

Chicken Game

	L	R
T	$0, \quad 0$	$101, \quad 1$
B	$1, 101$	$100, 100$

- $f_1(T) = f_2(L) = 0$ and $f_1(B) = f_2(R) = 1$.
- So B and R are the only security strategies.
- The outcome (B, R) can be justified using the concept of a security strategy.

Minmax

- Consider
$$F_i(s_{-i}) := \max_{s_i \in S_i} p_i(s_i, s_{-i}).$$
- We denote
$$minmax_i := \min_{s_{-i} \in S_{-i}} F_i(s_{-i}) = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} p_i(s_i, s_{-i}).$$
- Note For all $i \in \{1, \ldots, n\}$
$maxmin_i \leq minmax_i$.

Example

Consider

	L	M	R
T	3, –	4, –	5, –
B	6, –	2, –	1, –

Then

	L	M	R	f_1
T	3, –	4, –	5, –	3
B	6, –	2, –	1, –	1
F_1	6	4	5	

Here $maxmin_1 < minmax_1$.

Strictly Competitive Games

- Strictly competitive game: a two-player strategic game (S_1, S_2, p_1, p_2) such that for $i = 1, 2$ and $s, s' \in S_1 \times \cdots \times S_n$

$$p_i(s) \geq p_i(s') \text{ iff } p_{-i}(s) \leq p_{-i}(s').$$

- By negating we get

$$p_i(s) < p_i(s') \text{ iff } p_{-i}(s) > p_{-i}(s').$$

- Intuition: the interests of both players are diametrically opposed.

Example

	L	M	R
T	$3, 4$	$4, 3$	$5, 2$
B	$6, 0$	$2, 5$	$1, 6$

Zero-sum Games

- A two-player game is zero-sum if

$$p_1(s) + p_2(s) = 0.$$

- Example: Rock, Paper, Scissors game.
  - the rock defeats (breaks) scissors,
  - scissors defeat (cut) the paper,
  - the paper defeats (wraps) the rock.
- Reward matrix: only the payoffs of the first player are represented.

Example: Rock, Paper, Scissors Game

Reward matrix for the Rock, Paper, Scissors game:

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Characterization of Nash Equilibria

Theorem G a strictly competitive game.

(i) If for $i = 1, 2$, $maxmin_i = minmax_i$, then G has a Nash equilibrium.

(ii) If G has a Nash equilibrium, then for $i = 1, 2$, $maxmin_i = minmax_i$.

(iii) All Nash equilibria of G yield the same payoff, namely $maxmin_i$ for player i.

(iv) All Nash equilibria of G are of the form (s_1^*, s_2^*) where each s_i^* is a security strategy for player i.

Zero-sum Games

- Theorem G a zero-sum game. For $i = 1, 2$

$$maxmin_i = -minmax_{-i}$$

  and

$$minmax_i = -maxmin_{-i}.$$

- Zero-Sum Theorem For zero-sum games
  a Nash equilibrium exists iff $maxmin_1 = minmax_1$.
- When $maxmin_1 = minmax_1$ any pair of security strategies is called a saddle point and $maxmin_1$ is called the value of the game.

Example

- Take

	L	M	R
T	4	3	5
B	6	2	1

- We compute $maxmin_1$ and $minmax_1$:

	L	M	R	f_1
T	4	3	5	3
B	6	2	1	1
F_1	6	3	5	

- So $maxmin_1 = minmax_1 = 3$, which is the value of this game.
- (T, M) is the only pair of the security strategies, i.e., the only saddle point.

Minimax Theorem

- Note The mixed extension of a zero-sum game is a zero-sum game.
- Minimax Theorem (Von Neumann, 1928)
  Consider a finite zero-sum game $G := (S_1, S_2, p_1, p_2)$. Then for $i = 1, 2$

$$\min_{m_{-i} \in M_{-i}} \max_{m_i \in M_i} p_i(m_i, m_{-i}) = \max_{m_i \in M_i} \min_{m_{-i} \in M_{-i}} p_i(m_i, m_{-i}).$$

- Proof By Zero-sum Theorem and Note.