# Exercises

Each problem is worth 4 points. Please do at least 6 problems. Grades: 21 points = 5, 18 points = 4, 15 points = 3.

**Exercise 1** Which of the following 2-qubit states are entangled?

$$(a)\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle), \quad (b)\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad (c)\frac{1}{2}(|00\rangle + i|01\rangle - i|10\rangle + |11\rangle),$$

**Exercise 2** Find two different mixed states that have density matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$.

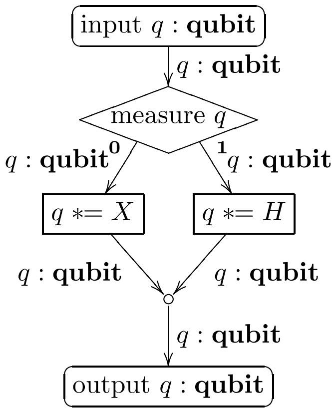
**Exercise 3** Recall that $D_n = \{A \in \mathbb{C}^{n\times n} \mid A \text{ is positive hermitian and } \operatorname{tr} A \leq 1\}$, with the partial order $A \sqsubseteq B$ iff $B - A$ is positive. Prove that every increasing sequence $A_1 \sqsubseteq A_2 \sqsubseteq \ldots$ has a least upper bound.

**Exercise 4** Recall

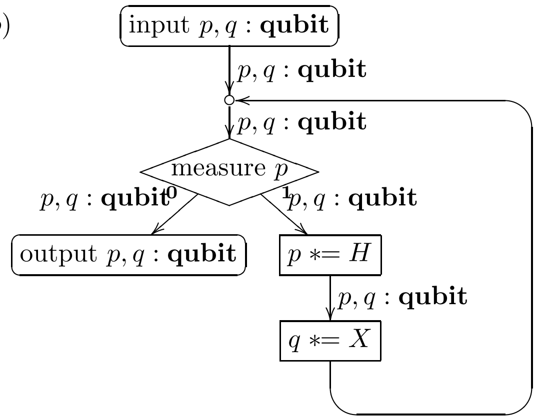
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad CNOT = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & X \end{array}\right), \quad H_c = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & H \end{array}\right).$$

We also use the notation $b \oplus\!= a$ to mean $a, b \mathrel{*}= CNOT$, i.e., on basis states: negate $b$ if $a = 1$. Compute the denotation of the following flow charts:

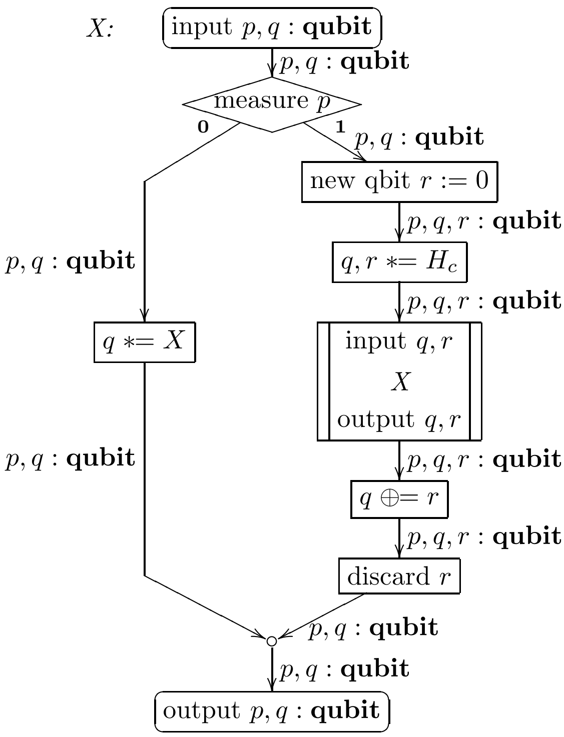
$(a)$




$(b)$




$(c)$



**Exercise 5** Prove that the following are equivalent for an $n \times n$-matrix $A$:

(a) $D$ is a density matrix, i.e., a hermitian positive matrix with $\operatorname{tr} D \leq 1$.

(b) There exist $m \geq 0$, unit vectors $v_1, \ldots, v_m$, and probabilities $p_1, \ldots, p_m$ with $p_1 + \ldots + p_m \leq 1$ such that $A = p_1 v_1 v_1^\dagger + \ldots + p_m v_m v_m^\dagger$.

**Exercise 6** Prove the theorems of Choi-Jamiołkowski and Kraus: the following are equivalent, for a linear function $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$:

(a) $F$ is completely positive.

(b) The characteristic matrix $\chi_F = \left(\begin{array}{c|c|c} F(E_{11}) & \cdots & F(E_{1n}) \\ \hline \vdots & \ddots & \vdots \\ \hline F(E_{n1}) & \cdots & F(E_{nn}) \end{array}\right)$ is positive.

(c) $F$ can can be written in the form $F(A) = \sum_i B_i A B_i^\dagger$, for some matrices $B_i$.

**Exercise 7** Which of the following are superoperators? For those that are, find a Kraus representation and a flow chart implementing them.

(a)

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

(b)

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$$

(c)

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \frac{d}{2} & \frac{b}{\sqrt{2}} \\ \frac{c}{\sqrt{2}} & \frac{d}{2} \end{pmatrix}$$

**Exercise 8** Consider the Pauli matrices $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, along with the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Prove:

(a) Every complex $2 \times 2$-matrix $U$ can be uniquely written in the form $\frac{1}{2}(wI + xX + yY + zZ)$, for complex numbers $w, x, y, z$.

(b) $U$ is hermitian if and only if $w, x, y, z$ are real.

(c) $\operatorname{tr} U = w$. In particular, $\operatorname{tr} U = 1$ if and only if $w = 1$.

(d) $\det U = w^2 - x^2 - y^2 - z^2$.

(e) $U$ is positive if and only if both its trace and determinant are $\geq 0$.

(f) Single-qubit mixed quantum states $U$ are in one-to-one correspondence with the closed unit ball $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$.

(g) Single-qubit pure quantum states are in one-to-one correspondence with the unit sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$.

(h) Prove: a mixed state is pure if and only if it cannot be written as a convex combination of distinct states.

This is known as the *Bloch sphere* representation of a qubit. It has a direct physical interpretation as the *spin* of a particle.