The Girard-Reynolds Isomorphism

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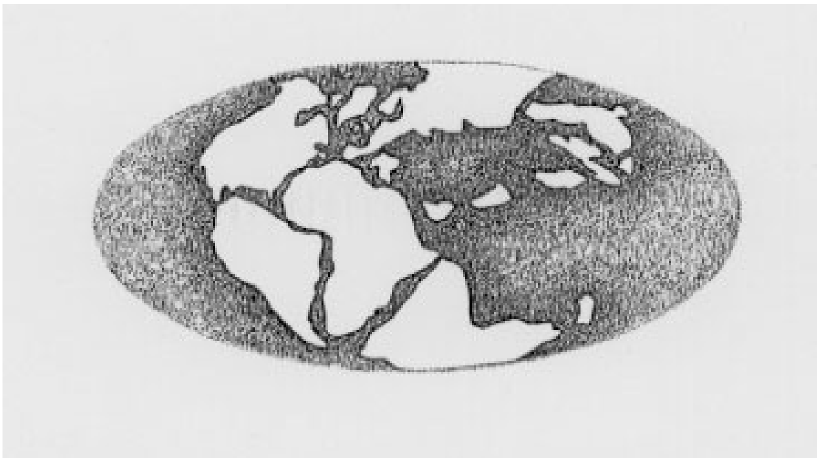
Coincidences

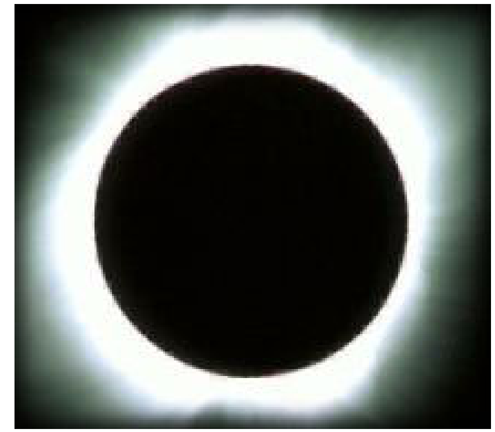
Curry-Howard

Hindley-Milner

Girard-Reynolds

Coincidences





A tale of Two Theorems

Girard's Representation Theorem

Reynolds's Abstraction Theorem

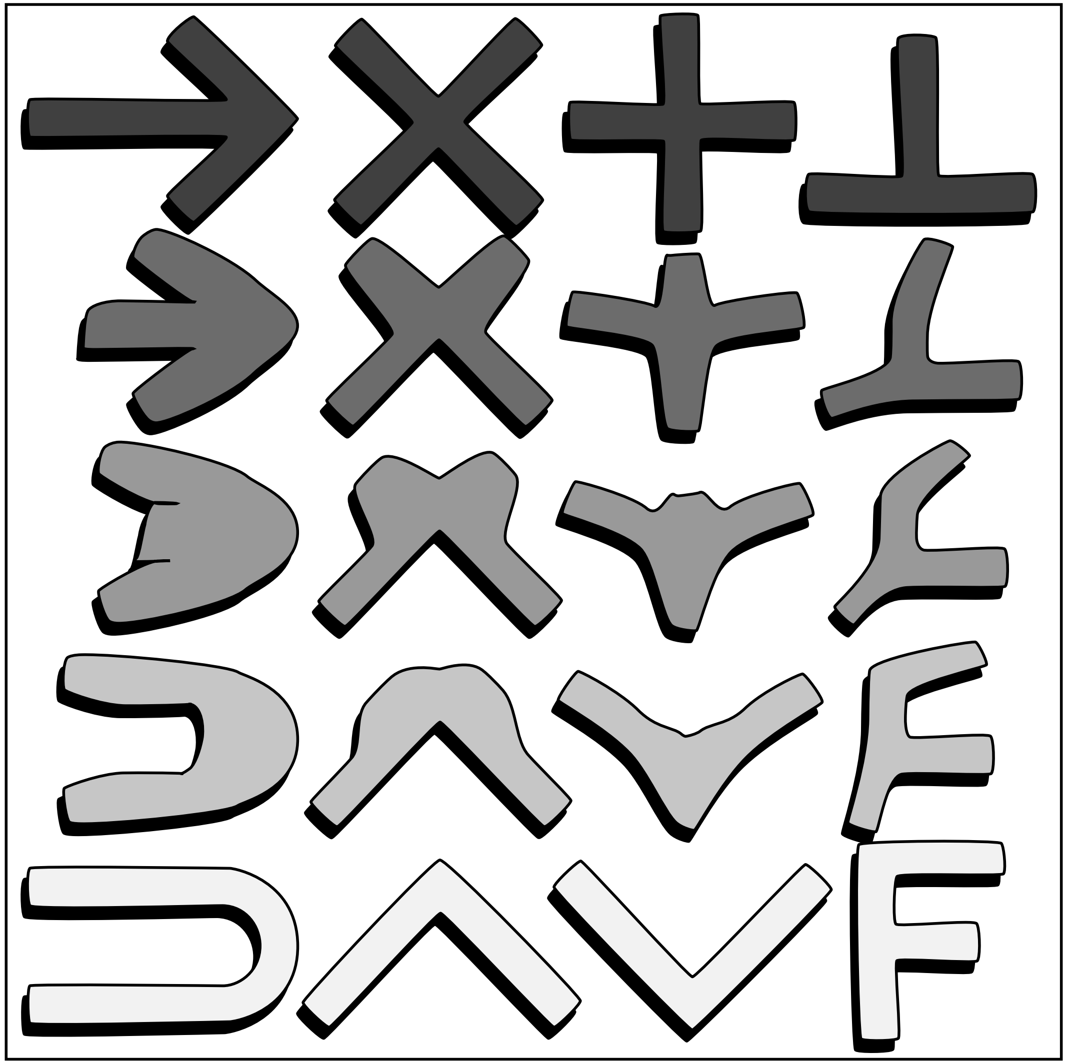
A tale of Two Theorems

Girard's Representation Theorem

projection : proofs $\longrightarrow$ terms

Reynolds's Abstraction Theorem

embedding : terms $\longrightarrow$ proofs



LC'90

The Curry-Howard homeomorphism

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \rightarrow \quad \times \quad + \quad \bot}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \longrightarrow \quad \times \quad + \quad \perp}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Girard-Reynolds Isomorphism

$$\frac{\forall^2 \quad \forall^1 \quad \longrightarrow}{\forall^2 \qquad\quad \longrightarrow}$$

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \to \quad \times \quad + \quad \bot}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Girard-Reynolds Isomorphism

$$\frac{\forall^2 \quad \forall^1 \quad \to}{\forall^2 \qquad\quad \to}$$

Rather than enriching the type systems to match logic, we impoverish logic to match the type structure.

— Daniel Leivant

Part I

Overview

The Girard projection

$$\mathsf{Nat} \;\equiv \{\, n \mid \forall X.\, (\forall m.\, m \in X \to \mathsf{S}\, m \in X) \to \mathsf{Z} \in X \to n \in X \,\}$$

$$\mathsf{Nat}^{\circ} \equiv \forall X.\, (X \to X) \to (X \to X)$$

The Reynolds embedding

$$\mathbf{Nat} \;\equiv \forall X.\, (X \to X) \to (X \to X)$$

$$\mathbf{Nat}^{*} \equiv \{\, n \mid \forall X.\, \forall s.\, (\forall m.\, m \in X \to s\, m \in X) \to \forall z.\, z \in X \to n\; s\; z \in X \,\}$$

Doubling and Parametricity

$$\mathbf{Nat} \equiv \forall X.\,(X \to X) \to (X \to X)$$

$$\begin{aligned}\mathbf{Nat}^* \equiv \{\, n \mid \forall X.\,\forall s.\,(\forall m.\, m \in X \to s\,m \in X) \to \\ \forall z.\, z \in X \to n\,s\,z \in X \,\}\end{aligned}$$

$$\begin{aligned}\mathbf{Nat}^{\approx} \equiv \{\, (n, n') \mid \forall X.\,\forall s, s'.\,(\forall m, m'.\,(m, m') \in X \to (s\,m, s'\,m') \in X) \to \\ \forall z, z'.\,(z, z') \in X \to (n\,s\,z, n'\,s'\,z') \in X \,\}\end{aligned}$$

$$\mathbf{Nat}^{=} \equiv \{\, (n, n') \mid n = n' \wedge n \in \mathbf{Nat}^* \,\}$$

Reynolds followed by Girard

$$\mathbf{Nat} \equiv \forall X.\,(X \to X) \to (X \to X)$$

$$\mathbf{Nat}^{*} \equiv \{\, n \,|\, \forall X.\, \forall s.\, (\forall m.\, m \in X \to s\, m \in X) \to \forall z.\, z \in X \to n\, s\, z \in X \,\}$$

$$\mathbf{Nat}^{*\circ} \equiv \forall X.\,(X \to X) \to (X \to X)$$

$$\mathbf{Nat}^{*\circ} \equiv \mathbf{Nat}$$

Girard followed by Reynolds

$$\mathsf{Nat} \equiv \{\, n \mid \forall X.\, (\forall m.\, m \in X \to \mathsf{S}\, m \in X) \to \mathsf{Z} \in X \to n \in X \,\}$$

$$\mathsf{Nat}^{\circ} \equiv \forall X.\, (X \to X) \to (X \to X)$$

$$\mathsf{Nat}^{\circ *} \equiv \{\, n \mid \forall X.\, \forall s.\, (\forall m.\, m \in X \to s\; m \in X) \to \forall z.\, z \in X \to n\; s\; z \in X \,\}$$

$$\mathbf{Nat}^{=} = \mathbf{Nat}^{\approx} \quad \text{iff} \quad \mathsf{Nat}^{\circ *} = \mathsf{Nat}$$

Part II

Examples

The Curry-Howard isomorphism (P2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash X \to X \qquad \Gamma \vdash X}{X \to X,\ X \vdash X}\ \to\text{-E}}{X \to X \vdash X \to X}\ \to\text{-I}}{\vdash (X \to X) \to X \to X}\ \to\text{-I}}{\vdash \forall X.\,(X \to X) \to X \to X}\ \forall^2\text{-I}$$

$$\Gamma \equiv X \to X,\quad X$$

The Curry-Howard isomorphism (F2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash s : X \to X \qquad \Gamma \vdash z : X}{s : X \to X,\, z : X \vdash s\ z : X}\ \to\text{-E}}{s : X \to X \vdash \lambda z^{X}.\, s\ z : X \to X}\ \to\text{-I}}{\vdash \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ z : (X \to X) \to X \to X}\ \to\text{-I}}{\vdash \Lambda X.\, \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ z : \forall X.\, (X \to X) \to X \to X}\ \forall^2\text{-I}$$

$$\Gamma \ \equiv\ s : X \to X,\, z : X$$

The Girard projection (P2)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \forall m.\, m \in X \to \mathsf{S}\, m \in X}{\Gamma \vdash \mathsf{Z} \in X \to \mathsf{S}\, \mathsf{Z} \in X}\ \forall^1\text{-E} \qquad \Gamma \vdash \mathsf{Z} \in X}{\forall m.\, m \in X \to \mathsf{S}\, m \in X,\ \ \mathsf{Z} \in X \vdash \mathsf{S}\, \mathsf{Z} \in X}\ \to\text{-E}}{\forall m.\, m \in X \to \mathsf{S}\, m \in X \vdash \mathsf{Z} \in X \to \mathsf{S}\, \mathsf{Z} \in X}\ \to\text{-I}}{\vdash (\forall m.\, m \in X \to \mathsf{S}\, m \in X) \to \mathsf{Z} \in X \to \mathsf{S}\, \mathsf{Z} \in X}\ \to\text{-I}}{\vdash \forall X.\, (\forall m.\, m \in X \to \mathsf{S}\, m \in X) \to \mathsf{Z} \in X \to \mathsf{S}\, \mathsf{Z} \in X}\ \forall^2\text{-I}
$$

$$
\Gamma \equiv \forall m.\, m \in X \to \mathsf{S}\, m \in X, \quad \mathsf{Z} \in X
$$

The Girard projection (F2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash s : X \to X \qquad \Gamma \vdash z : X}{s : X \to X,\ z : X \vdash s\ z : X}\ \to\text{-E}}{s : X \to X \vdash \lambda z^{X}.\, s\ z : X \to X}\ \to\text{-I}}{\vdash \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ z : (X \to X) \to X \to X}\ \to\text{-I}}{\vdash \Lambda X.\, \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ z : \forall X.\, (X \to X) \to X \to X}\ \forall^2\text{-I}$$

$$\Gamma \equiv s : X \to X,\ z : X$$

The Reynolds embedding (F2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash s : X \to X \qquad \Gamma \vdash z : X}{s : X \to X,\ z : X \vdash s\ z : X}\ \to\text{-E}}{s : X \to X \vdash \lambda z^{X}.\, s\ z : X \to X}\ \to\text{-I}}{\vdash \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ z : (X \to X) \to X \to X}\ \to\text{-I}}{\vdash \Lambda X.\, \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ z : \forall X.\, (X \to X) \to X \to X}\ \forall^2\text{-I}$$

$$\Gamma \ \equiv\ s : X \to X,\ z : X$$

The Reynolds embedding (P2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \forall m.\, m \in X \to s\, m \in X}{\Gamma \vdash z \in X \to s\, z \in X}\ \forall^1\text{-E} \qquad \Gamma \vdash z \in X}{\forall m.\, m \in X \to s\, m \in X,\ z \in X \vdash s\, z \in X}\ \to\text{-E}}{\forall m.\, m \in X \to s\, m \in X \vdash z \in X \to s\, z \in X}\ \to\text{-I}}{\forall m.\, m \in X \to s\, m \in X \vdash \forall z.\, z \in X \to (\lambda z.\, s\, z)\, z \in X}\ \forall^1\text{-I}}{\vdash (\forall m.\, m \in X \to s\, m \in X) \to \forall z.\, z \in X \to (\lambda z.\, s\, z)\, z \in X}\ \to\text{-I}}{\vdash \forall s.\, (\forall m.\, m \in X \to s\, m \in X) \to \forall z.\, z \in X \to (\lambda s.\, \lambda z.\, s\, z)\, s\, z \in X}\ \forall^1\text{-I}}{\vdash \forall X.\, \forall s.\, (\forall m.\, m \in X \to s\, m \in X) \to \forall z.\, z \in X \to (\lambda s.\, \lambda z.\, s\, z)\, s\, z \in X}\ \forall^2\text{-I}$$

$$\Gamma \;\equiv\; \forall m.\, m \in X \to s\, m \in X,\ z \in X$$

Doubling (P2)

$$\Gamma \vdash \forall \overset{m}{m'}.\, \overset{m}{m'} \in X \to \overset{s\,m}{s'\,m'} \in X$$

$\forall^1$-E

$$\Gamma \vdash \overset{z}{z'} \in X \to \overset{s\,z}{s'\,z'} \in X \qquad\qquad \Gamma \vdash \overset{z}{z'} \in X$$

$\to$-E

$$\forall \overset{m}{m'}.\, m \in X \to \overset{s\,m}{s'\,m'} \in X,\ \overset{z}{z'} \in X \vdash \overset{s\,z}{s'\,z'} \in X$$

$\to$-I

$$\forall \overset{m}{m'}.\, \overset{m}{m'} \in X \to \overset{s\,m}{s'\,m'} \in X \vdash \overset{z}{z'} \in X \to \overset{s\,z}{s'\,z'} \in X$$

$\forall^1$-I

$$\forall \overset{m}{m'}.\, \overset{m}{m'} \in X \to \overset{s\,m}{s'\,m'} \in X \vdash \forall \overset{z}{z'}.\, \overset{z}{z'} \in X \to \overset{(\lambda z.\, s\ z)\ z}{(\lambda z'.\, s'\ z')\ z'} \in X$$

$\to$-I

$$\vdash \left(\forall \overset{m}{m'}.\, \overset{m}{m'} \in X \to \overset{s\,m}{s'\,m'} \in X\right) \to \forall \overset{z}{z'}.\, \overset{z}{z'} \in X \to \overset{(\lambda z.\, s\ z)\ z}{(\lambda z'.\, s'\ z')\ z'} \in X$$

$\forall^1$-I

$$\vdash \forall \overset{s}{s'}.\, \left(\forall \overset{m}{m'}.\, \overset{m}{m'} \in X \to \overset{s\,m}{s'\,m'} \in X\right) \to \forall \overset{z}{z'}.\, \overset{z}{z'} \in X \to \overset{(\lambda s.\, \lambda z.\, s\ z)\ s\ z}{(\lambda s'.\, \lambda z'.\, s'\ z')\ s'\ z'} \in X$$

$\forall^2$-I

$$\vdash \forall X.\, \forall \overset{s}{s'}.\, \left(\forall \overset{m}{m'}.\, \overset{m}{m'} \in X \to \overset{s\,m}{s'\,m'} \in X\right) \to \forall \overset{z}{z'}.\, \overset{z}{z'} \in X \to \overset{(\lambda s.\, \lambda z.\, s\ z)\ s\ z}{(\lambda s'.\, \lambda z'.\, s'\ z')\ s'\ z'} \in X$$

$$\Gamma \;\equiv\; \forall \overset{m}{m'}.\, m \in X \to \overset{s\,m}{s'\,m'} \in X,\ \overset{z}{z'} \in X$$

Successor (P2)

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{
\dfrac{\Theta \vdash \phi_s}{\Theta \vdash n \in X \to \mathsf{S}\; n \in X}\;\forall^1\text{-E}
\qquad
\dfrac{\dfrac{\Theta \vdash n \in \mathsf{Nat}}{\Theta \vdash \phi_s \to \phi_z \to n \in X}\;\forall^2\text{-E} \quad \Theta \vdash \phi_s \quad \Theta \vdash \phi_z}{\Theta \vdash n \in X}\;\to\text{-E}^2
}{n \in \mathsf{Nat}, \phi_s, \phi_z \vdash \mathsf{S}\; n \in X}\;\to\text{-E}
}{n \in \mathsf{Nat} \vdash \phi_s \to \phi_z \to \mathsf{S}\; n \in X}\;\to\text{-I}^2
}{n \in \mathsf{Nat} \vdash \mathsf{S}\; n \in \mathsf{Nat}}\;\forall^2\text{-I}
}{\vdash n \in \mathsf{Nat} \to \mathsf{S}\; n \in \mathsf{Nat}}\;\to\text{-I}
$$

$$
\begin{aligned}
\phi_s &\equiv \forall m.\, m \in X \to \mathsf{S}\; m \in X \\
\phi_z &\equiv \mathsf{Z} \in X \\
\Theta &\equiv n \in \mathsf{Nat},\ \phi_s,\ \phi_z
\end{aligned}
$$

Successor (F2)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \Gamma \vdash s : X \to X
        \qquad
        \dfrac{
          \dfrac{\Gamma \vdash n : \mathbf{Nat}}{\Gamma \vdash n\ X : A_s \to A_z \to X}\ \forall^2\text{-E}
          \qquad
          \Gamma \vdash s : A_s
          \quad
          \Gamma \vdash z : A_z
        }{\Gamma \vdash n\ X\ s\ z : X}\ \to\text{-E}^2
      }{n : \mathbf{Nat},\ s : A_s,\ z : A_z \vdash s\ (n\ X\ s\ z) : X}\ \to\text{-E}
    }{n : \mathbf{Nat} \vdash \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z) : A_s \to A_z \to X}\ \to\text{-I}^2
  }{n : \mathbf{Nat} \vdash \Lambda X.\, \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z) : \mathbf{Nat}}\ \forall^2\text{-I}
}{\vdash \lambda n^{\mathbf{Nat}}.\, \Lambda X.\, \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z) : \mathbf{Nat} \to \mathbf{Nat}}\ \to\text{-I}
$$

$$
\begin{aligned}
A_s &\equiv X \to X \\
A_z &\equiv X \\
\Gamma &\equiv n : \mathbf{Nat},\ s : X \to X,\ z : X
\end{aligned}
$$

Part III

Details

Second-order lambda calculus (F2)

$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i}\ \text{Id}$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x{:}A.\, u : A \to B}\ \to\text{-I} \qquad \frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t : A}{\Gamma \vdash s\, t : B}\ \to\text{-E}$$

$$\frac{\Gamma \vdash u : B}{\Gamma \vdash \Lambda X.\, u : \forall X.\, B}\ \forall^2\text{-I} \quad (X \text{ not free in } \Gamma) \qquad \frac{\Gamma \vdash s : \forall X.\, B}{\Gamma \vdash s\, A : B[X := A]}\ \forall^2\text{-E}$$

Second-order propositional logic (P2)

$$\frac{}{\phi_1, \ldots, \phi_n \vdash \phi_i}\ \text{Id}$$

$$\frac{\Theta, \phi \vdash \psi}{\Theta \vdash \phi \to \psi}\ {\to}\text{-I} \qquad \frac{\Theta \vdash \phi \to \psi \qquad \Theta \vdash \phi}{\Theta \vdash \psi}\ {\to}\text{-E}$$

$$\frac{\Theta \vdash \psi}{\Theta \vdash \forall x.\, \psi}\ \forall^1\text{-I} \quad (x \text{ not free in } \Theta) \qquad \frac{\Theta \vdash \forall x.\, \psi}{\Theta \vdash \psi[x := M]}\ \forall^1\text{-E}$$

$$\frac{\Theta \vdash \psi}{\Theta \vdash \forall X.\, \psi}\ \forall^2\text{-I} \quad (X \text{ not free in } \Theta) \qquad \frac{\Theta \vdash \forall X.\, \psi}{\Theta \vdash \psi[X := \{\, x \mid \phi \,\}]}\ \forall^2\text{-E}$$

$$\frac{(\Theta \vdash \psi)\ [x := M]}{(\Theta \vdash \psi)\ [x := N]}\ \beta \quad (M =_\beta N)$$

The Reynolds embedding

Terms

$$
\begin{aligned}
|x| &\equiv x \\
|\lambda x{:}A.\, u| &\equiv \lambda x.\, |u| \\
|s\ t| &\equiv |s|\ |t| \\
|\Lambda X.\, u| &\equiv |u| \\
|s\ A| &\equiv |s|
\end{aligned}
$$

Types

$$
\begin{aligned}
X^* &\equiv \{\, z \mid z \in X \,\} \\
(A \to B)^* &\equiv \{\, z \mid \forall x.\, x \in A^* \to z\ x \in B^* \,\} \\
(\forall X.\, B)^* &\equiv \{\, z \mid \forall X.\, z \in B^* \,\}
\end{aligned}
$$

The Reynolds Embedding, continued

$$\left(\frac{}{\Gamma,\, x : A \vdash x : A}\,\mathrm{Id}\right)^* \equiv \frac{}{\Gamma^*,\, x \in A^* \vdash x \in A^*}\,\mathrm{Id}$$

$$\left(\frac{\Gamma,\, x : A \vdash u : B}{\Gamma \vdash \lambda x.\, u : A \to B} \to\text{-I}\right)^* \equiv \dfrac{\dfrac{\dfrac{\Gamma^*,\, x \in A^* \vdash |u| \in B^*}{\Gamma^*,\, x \in A^* \vdash (\lambda x.\, |u|)\; x \in B^*}\,\beta}{\Gamma^* \vdash x \in A^* \to (\lambda x.\, |u|)\; x \in B^*} \to\text{-I}}{\Gamma^* \vdash \forall x.\, x \in A^* \to (\lambda x.\, |u|)\; x \in B^*}\,\forall^1\text{-I}$$

$$\left(\frac{\Gamma \vdash u : B}{\Gamma \vdash \Lambda X.\, u : \forall X.\, B}\,\forall^2\text{-I}\right)^* \equiv \frac{\Gamma^* \vdash |u| \in B^*}{\Gamma^* \vdash \forall X.\, |u| \in B^*}\,\forall^2\text{-I}$$

The Girard projection

$$
\begin{aligned}
(M \in X)^\circ &\equiv X \\
(\phi \longrightarrow \psi)^\circ &\equiv \phi^\circ \longrightarrow \psi^\circ \\
(\forall x.\, \psi)^\circ &\equiv \psi^\circ \\
(\forall X.\, \psi)^\circ &\equiv \forall X.\, \psi^\circ
\end{aligned}
$$

The Girard projection, continued

$$\left(\frac{}{\Theta,\ \phi \vdash \phi}\ \text{Id}\right)^{\circ} \quad \equiv \quad \frac{}{\Theta^{\circ},\ z : \phi^{\circ} \vdash z : \phi^{\circ}}\ \text{Id}$$

$$\left(\frac{\Theta,\ \phi \vdash \psi}{\Theta \vdash \phi \to \psi}\ {\to}\text{-I}\right)^{\circ} \quad \equiv \quad \frac{\Theta^{\circ},\ z : \phi^{\circ} \vdash u : \psi^{\circ}}{\Theta^{\circ} \vdash \lambda z{:}\phi^{\circ}.\, u : \phi^{\circ} \to \psi^{\circ}}\ {\to}\text{-I}$$

$$\left(\frac{\Theta \vdash \psi}{\Theta \vdash \forall x.\, \psi}\ \forall^{1}\text{-I}\right)^{\circ} \quad \equiv \quad \Theta^{\circ} \vdash u : \psi^{\circ}$$

$$\left(\frac{\Theta \vdash \psi}{\Theta \vdash \forall X.\, \psi}\ \forall^{2}\text{-I}\right)^{\circ} \quad \equiv \quad \frac{\Theta^{\circ} \vdash u : \psi^{\circ}}{\Theta^{\circ} \vdash \Lambda X.\, u : \forall X.\, \psi^{\circ}}\ \forall^{2}\text{-I}$$

Doubling

$$
\begin{aligned}
(M \in X)^{\ddagger} &\equiv (M, M') \in X \\
(\phi \to \psi)^{\ddagger} &\equiv \phi^{\ddagger} \to \psi^{\ddagger} \\
(\forall x.\, \psi)^{\ddagger} &\equiv \forall x.\, \forall x'.\, \psi^{\ddagger} \\
(\forall X.\, \psi)^{\ddagger} &\equiv \forall X.\, \psi^{\ddagger}
\end{aligned}
$$

Abstraction Theorem

$$x_1 : A_1, \ldots, x_n : A_n \vdash t : B \qquad \text{in F2}$$

$$\Rightarrow$$

$$(x_1, x_1') \in A_1^{*\ddagger}, \ldots, (x_n, x_n') \in A_n^{*\ddagger} \vdash (|t|, |t|') \in B^{*\ddagger} \qquad \text{in P2}$$

Equality

Leibniz equality

$$M = N \quad \equiv \quad \forall Z.\, M \in Z \to N \in Z$$

Identity relation

$$A^{=} \quad \equiv \quad \{\, (z, z') \mid z = z' \wedge z \in A^{*} \,\}$$

Parametric closure

$$A^{\approx} \quad \equiv \quad A^{*\ddagger}[X_1^{\ddagger} := X_1^{=}, \ldots, X_1^{\ddagger} := X_1^{=}]$$

where $X_1, \ldots, X_n$ are the free type variables in A

Breaking the proof into parts

$$\mathbf{Nat}^{\approx} \subseteq \mathbf{Nat}^{=}$$

$$\mathsf{Nat} \subseteq \mathbf{Nat}^{*}$$

$$\mathbf{Nat}^{*} \subseteq \mathsf{Nat} \quad \text{implies} \quad \mathbf{Nat}^{=} \subseteq \mathbf{Nat}^{\approx}$$

$$\mathbf{Nat}^{=} \subseteq \mathbf{Nat}^{\approx} \quad \text{implies} \quad \mathbf{Nat}^{*} \subseteq \mathsf{Nat}$$

Böhm and Berarducci's Lemma

$$(z, z) \in \mathbf{Nat}^{\approx} \quad \text{implies} \quad z\,\mathsf{S}\,\mathsf{Z} = z$$

Girard-Reynolds Isomorphism

$$\mathbf{Nat}^{=} = \mathbf{Nat}^{\approx} \quad \text{iff} \quad \mathbf{Nat}^{*} = \mathsf{Nat}$$

A part of the proof

$$n \in \mathbf{Nat}^*$$

$\Rightarrow$ (definition deductive naturals, Reynolds embedding)

$$\forall X. \forall s. (\forall m. m \in X \to s\, m \in X) \to \forall z. z \in X \to n\, s\, z \in X$$

$\Rightarrow$ (instantiate $X := \mathsf{Nat}$, $s := \mathsf{S}$, $z := \mathsf{Z}$)

$$(\forall m. m \in \mathsf{Nat} \to \mathsf{S}\, m \in \mathsf{Nat}) \to \mathsf{Z} \in \mathsf{Nat} \to n\, \mathsf{S}\, \mathsf{Z} \in \mathsf{Nat}$$

$\Rightarrow$ (Constructor Lemma)

$$n\, \mathsf{S}\, \mathsf{Z} \in \mathsf{Nat}$$

$\Rightarrow$ (parametricity of naturals, Böhm and Berarducci's Lemma)

$$n \in \mathsf{Nat}$$

Part IV

Conclusion

Related work

Girard

Reynolds

Böhm and Beararducci

Leivant

Krivine and Parigot

Mairson

Reynolds and Plotkin

Bainbridge, Freyd, Scedrov, and Scott

Hasegawa

Abadi, Cardelli, Curien, and Plotkin

Pitts

Takeuti

Conclusion

My most recent work has been with formal modelling of XML.

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This work has been depressingly popular.

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This work has been depressingly popular.

Fortunately, Girard and Reynolds will be remembered long after XML is forgotten.