FEDOR V. FOMIN

Graph Minors, Bidimensionality and Algorithms

PART II

Warsaw, 2010

PART II

Branchwidth and grids

WIN/WIN approach

Tree-likeness

We have to define the tree-likeness of a graph.

Branchwidth is a tree-likeness measure, alternative to treewidth,

appeared in GM-X (1991).

Main tool: Branch Decompositions

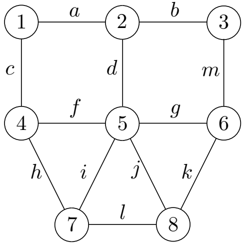
Definition

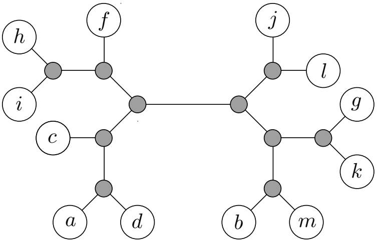
A branch decomposition of a graph $G = (V, E)$ is a tuple (T, μ)

where

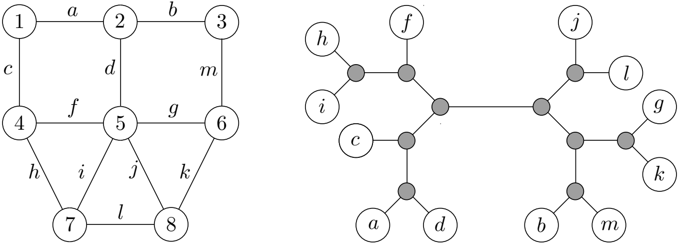
- T is a tree with degree 3 for all internal nodes.
- μ is a bijection between the leaves of T and $E(G)$.

Example of Branch Decomposition

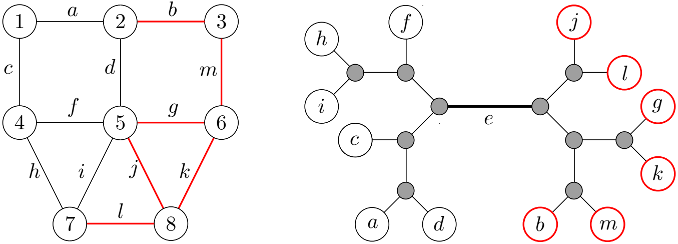




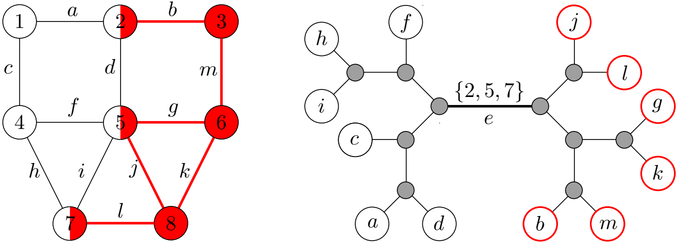
Edge $e \in T$ partitions the edge set of G in A_e and B_e



Edge $e \in T$ partitions the edge set of G in A_e and B_e



Middle set $\mathrm{mid}(e) = V(A_e) \cap V(B_e)$



Branchwidth

- The *width* of a branch decomposition is $\max_{e \in T} |\mathrm{mid}(e)|$.
- The *branchwidth* of a graph G is the minimum width over all branch decompositions of G.

Branchwidth vs Treewidth

Lemma (Robertson-Seymour)

For every graph G,

$$\mathrm{branchwidth}(G) \leq \mathrm{treewidth}(G) + 1 \leq \lfloor \tfrac{3}{2}\mathrm{branchwidth}(G) \rfloor.$$

Exercises

- What is the branchwidth of a tree?
- Complete graph on n vertices?
- $(\ell \times \ell)$-grid?
- Prove Treewidth vs Branchwidth lemma

Small Branchwidth is good for designing algorithms!

Small Branchwidth is good for designing algorithms!

Theorem (Courcelle)

Any MSOL expressible property can be decided in linear time for graphs of bounded branchwidth.

Advantage of Courcelle's Theorem: It constructs the algorithm

Advantage of Courcelle's Theorem: It constructs the algorithm

Drawback of Courcelle's Theorem: the contribution of the formula and the branchwidth in the running time is immense.

Advantage of Courcelle's Theorem: It constructs the algorithm

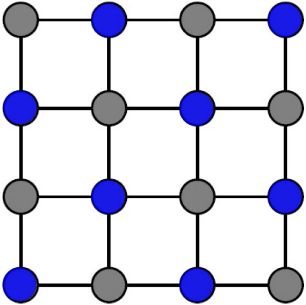
Drawback of Courcelle's Theorem: the contribution of the formula and the branchwidth in the running time is immense.

What do we do for specific problems?

Standard (or, not so standard) dynamic programming!

Vertex Cover

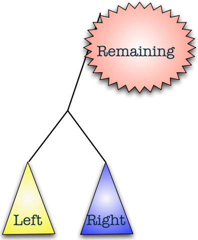
A *vertex cover* C of a graph G, $vc(G)$, is a set of vertices such that every edge of G has at least one endpoint in C.

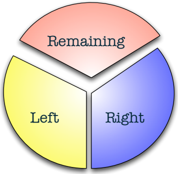


Dynamic programming: Vertex Cover

Main idea—dynamic programming.

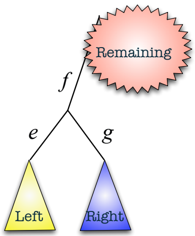
- Start from leaves, compute all possible vertex covers of each edge
- We have two branches Left and Right

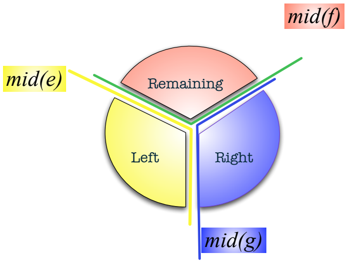




Dynamic programming: Vertex Cover

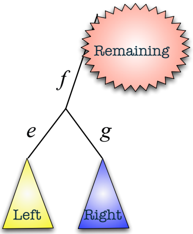
- $\text{mid}(e) = V(\text{Left}) \cap (V(\text{Right}) \cup V(\text{Remaining}))$
- $\text{mid}(g) = V(\text{Right}) \cap (V(\text{Left}) \cup V(\text{Remaining}))$
- $\text{mid}(f) = V(\text{Remaining}) \cap (V(\text{Left}) \cup V(\text{Right}))$

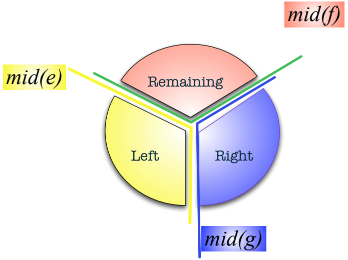




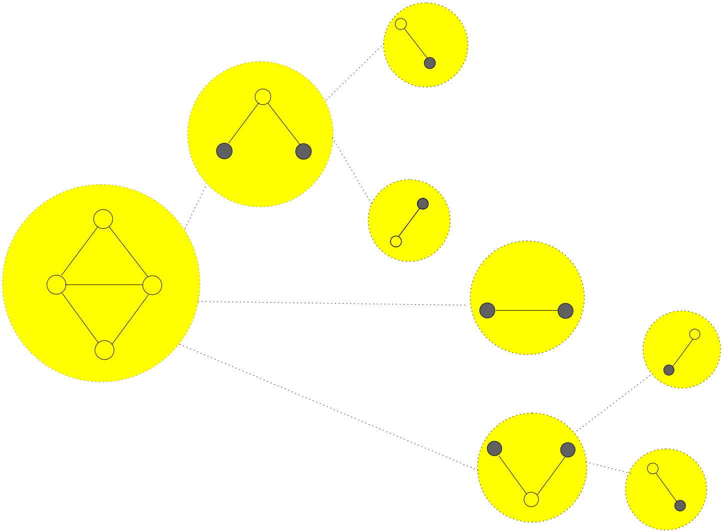
Dynamic programming: Vertex Cover

- For every $A \subseteq \mathrm{mid}(f)$ we want to compute a minimum size c_A of vertex cover C_A in $\mathrm{Left} \cup \mathrm{Right}$ such that $C_A \cap \mathrm{mid}(f) = A$
- $c_A = \min_{\substack{B \subseteq \mathrm{mid}(e) \\ C \subseteq \mathrm{mid}(g) \\ (B \cup C) \cap \mathrm{mid}(f) = A}} c_B + c_C - |B \cap C|$





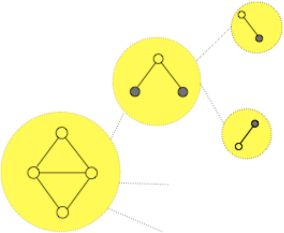
Dynamic programming: Vertex Cover

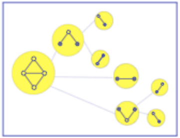


Dynamic programming: Vertex Cover

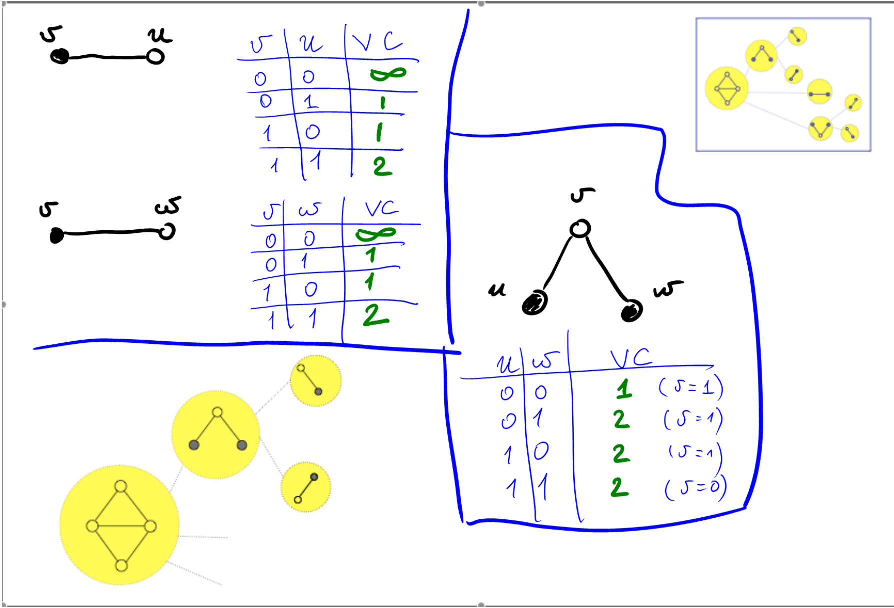
v	u	VC
0	0	∞
0	1	1
1	0	1
1	1	2

v	w	VC
0	0	∞
0	1	1
1	0	1
1	1	2





Dynamic programming: Vertex Cover

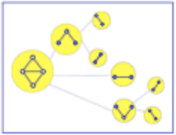


v	u	VC
0	0	∞
0	1	1
1	0	1
1	1	2

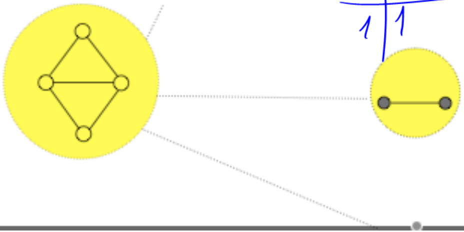
v	w	VC
0	0	∞
0	1	1
1	0	1
1	1	2

u	w	VC	
0	0	1	(v=1)
0	1	2	(v=1)
1	0	2	(v=1)
1	1	2	(v=0)

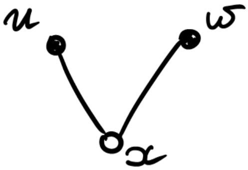
Dynamic programming: Vertex Cover



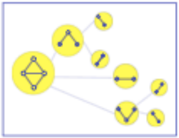
u	w	VC
0	0	∞
0	1	1
1	0	1
1	1	2

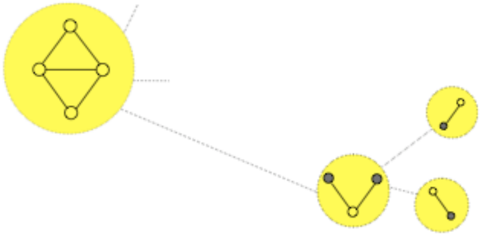


Dynamic programming: Vertex Cover

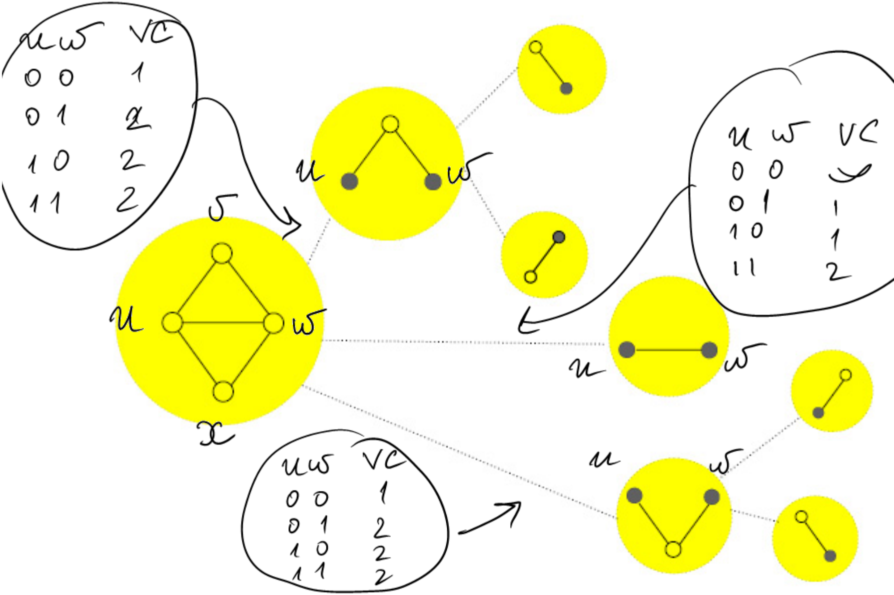


u w	VC
0 0	1
0 1	2
1 0	2
1 1	2

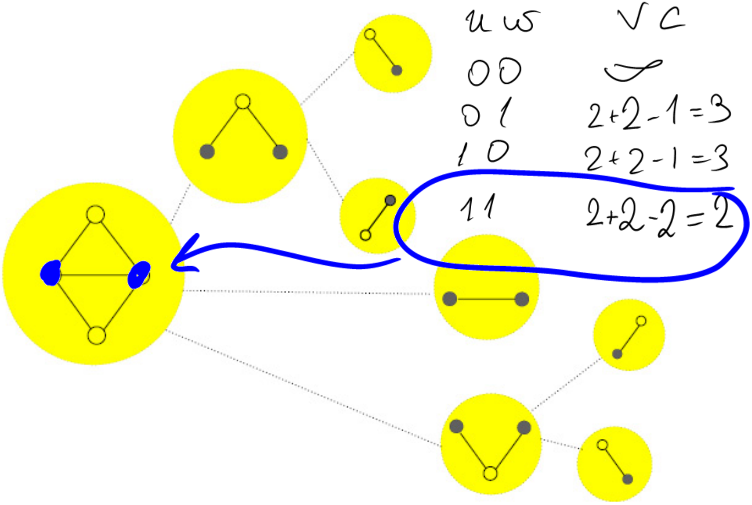




Dynamic programming: Vertex Cover



Dynamic programming: Vertex Cover



Dynamic programming: Vertex Cover

Let $\ell = \mathbf{bw}(G)$ and $m = |E(G)|$.

- Running time: size of every table for middle set is $O(2^{\ell})$.
- To compute a new table: $O(2^{2\ell})$
- Number of steps $O(m)$
- Total running time: $O(2^{2\ell}m)$.

Dynamic programming: Vertex Cover

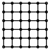
Exercise

Try to improve the running time, say to $O(2^{1.5\ell}m)$.

Dynamic programming: Counting Matchings

Grid Theorem

Theorem (Robertson, Seymour & Thomas, 1994)

Let $\ell \geq 1$ be an integer. Every planar graph of branchwidth $\geq 4\ell$ contains  $_{\ell}$ *as a minor.*

Grid Theorem: Sketch of the proof

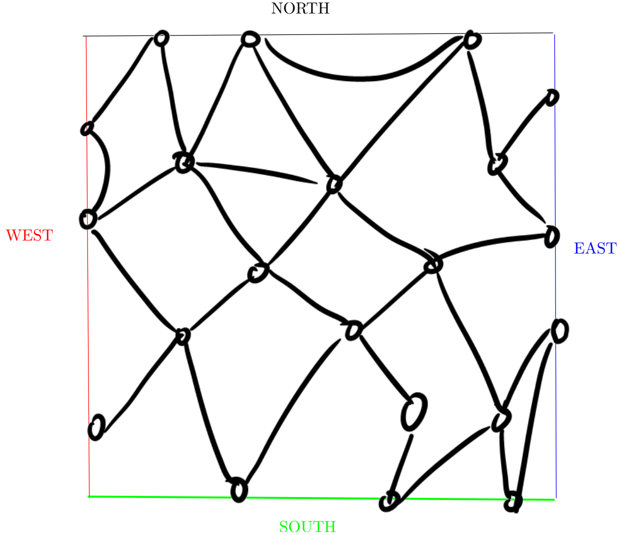
The proof is based on Menger's Theorem

Theorem (Menger 1927)

Let G be a finite undirected graph and x and y two nonadjacent vertices. The size of the minimum vertex cut for x and y (the minimum number of vertices whose removal disconnects x and y) is equal to the maximum number of pairwise vertex-disjoint paths from x to y.

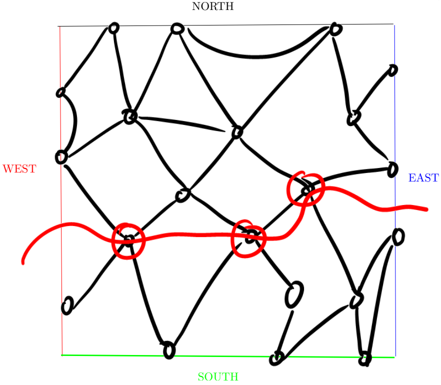
Grid Theorem: Sketch of the proof

Let G be a plane graph that has no $(\ell \times \ell)$-grid as a minor.



Grid Theorem: Sketch of the proof

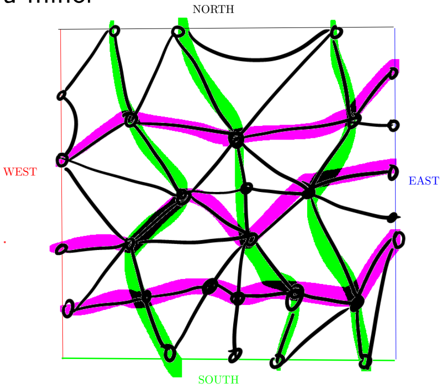
Either East can be separated from West, or South from North by



removing at most ℓ vertices

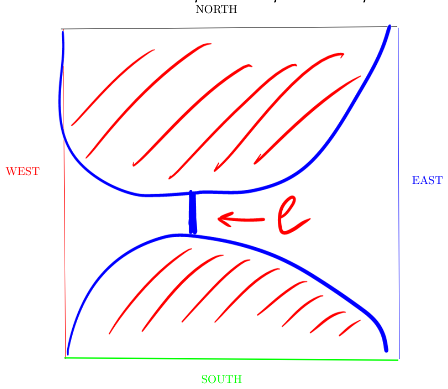
Grid Theorem: Sketch of the proof

Otherwise by making use of Menger we can construct $\ell \times \ell$ grid as a minor

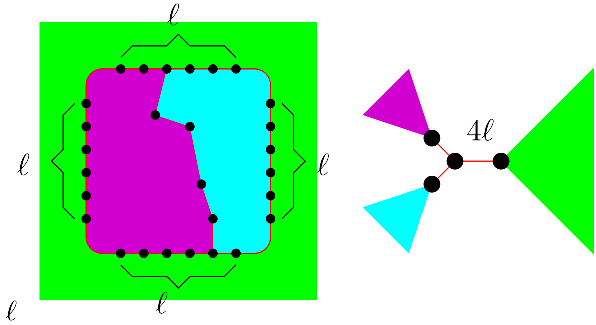


Grid Theorem: Sketch of the proof

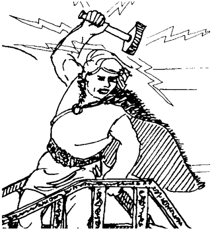
Partition the edges. Every time the middle set contains only vertices of East, West, South, and North, at most 4ℓ in total.



Grid Theorem: Sketch of the proof



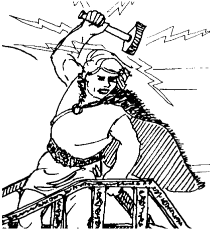
We have the hammer!



WIN/WIN on planar graphs:

Either small branch-width or large grid as a minor

We have the hammer!



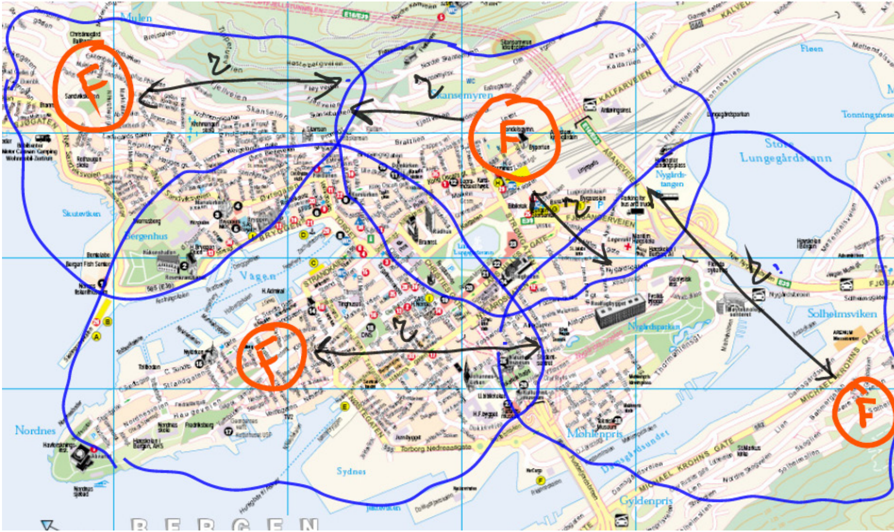
WIN/WIN on planar graphs:

Either small branch-width or large grid as a minor

APPLICATION I: Parameterized Algorithms

Example: The city of Bergen

How to place k fire stations such that every building is within r city blocks from the nearest fire station?



How to place k fire stations?

- Some simplifications: Bergen is a planar graph and $r = 1$.

How to place k fire stations?

- Some simplifications: Bergen is a planar graph and $r = 1$.
- There is a linear kernel $O(k)$ for dominating set on planar graph, so $2^{O(k)}n^{O(1)}$ algorithm is possible

How to place k fire stations?

- Some simplifications: Bergen is a planar graph and $r = 1$.
- There is a linear kernel $O(k)$ for dominating set on planar graph, so $2^{O(k)}n^{O(1)}$ algorithm is possible
- We show how to get *subexponential* $2^{O(\sqrt{k})}n^{O(1)}$ algorithms.

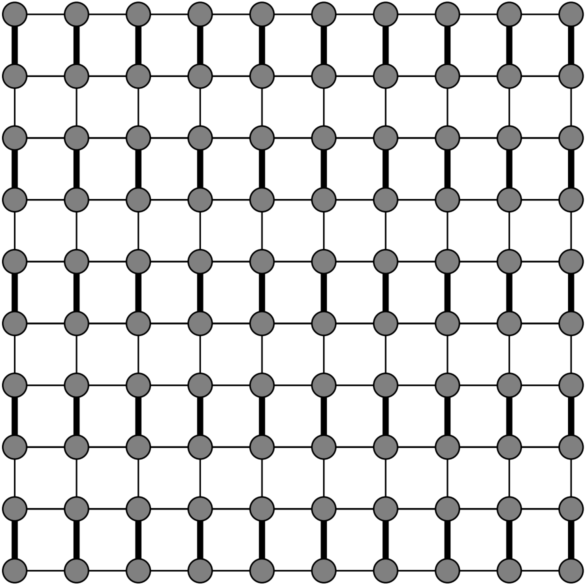
How to place k fire stations?

- Some simplifications: Bergen is a planar graph and $r = 1$.
- There is a linear kernel $O(k)$ for dominating set on planar graph, so $2^{O(k)}n^{O(1)}$ algorithm is possible
- We show how to get *subexponential* $2^{O(\sqrt{k})}n^{O(1)}$ algorithms.
- The idea works even when Bergen has more complicated structure, like embedded on a surface of bounded genus, or excluding some fixed graph as a minor; it works for every fixed $r \geq 1$, and for many other problems

How to compute branchwidth

- NP-hard in general (Seymour-Thomas, Combinatorica 1994)
- On planar graphs can be computed in time $O(n^3)$ (Seymour-Thomas, Combinatorica 1994 and Gu-Tamaki, ICALP 2005)
- RST grid theorem provides 4-approximation on planar graphs.
- On general graphs there are constant factor approximation algorithms of running time $2^{O(\mathbf{bw}(G))}n^{O(1)}$

We know enough to solve Vertex Cover!



$\mathbf{vc}(H_{r,r}) \geq \frac{r^2}{2}$

We know enough to solve Vertex Cover!

Let G be a planar graph of

branchwidth $\geq \ell$

We know enough to solve Vertex Cover!

Let G be a planar graph of branchwidth $\geq \ell$ $\Longrightarrow$ G contains an $(\ell/4 \times \ell/4)$-grid H as a minor

We know enough to solve Vertex Cover!

Let G be a planar graph of branchwidth $\geq \ell$ $\Longrightarrow$ G contains an $(\ell/4 \times \ell/4)$-grid H as a minor

The size of any vertex cover of H is at least $\ell^2/32$. Since H is a minor of G, the size of any vertex cover of G is at least $\ell^2/32$.

We know enough to solve Vertex Cover!

Let G be a planar graph of branchwidth $\geq \ell$ $\Longrightarrow$ G contains an $(\ell/4 \times \ell/4)$-grid H as a minor

The size of any vertex cover of H is at least $\ell^2/32$. Since H is a minor of G, the size of any vertex cover of G is at least $\ell^2/32$.

WIN/WIN

If $k < \ell^2/32$, we say "NO"

If $k \geq \ell^2/32$, then we do DP in time $O(2^{2\ell}m) = O(2^{O(\sqrt{k})}m)$.

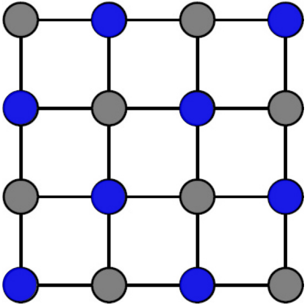
Challenges to discuss

- How to generalize the idea to work for other parameters?
- Does not work for Dominating Set. Why?
- Is planarity essential?
- Dynamic programming. Does MSOL helps here?

Parameters (Reminder)

- *Parameter* P is a function mapping graphs to nonnegative integers.
- The *parameterized problem* associated with P asks, for some fixed k, whether for a given graph G, $P(G) \leq k$ (for minimization) and $P(G) \geq k$ (for maximization problem).
- A parameter P is *closed* under taking of minors/contractions (or, briefly, minor/contraction closed) if for every graph H, $H \preceq G$ / $H \preceq_c G$ implies that $P(H) \leq P(G)$.

k-Vertex Cover

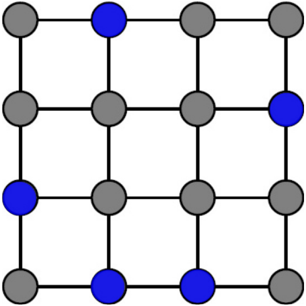


k-Vertex Cover is closed under taking minors.

Examples of parameters: k-Dominating set

A *dominating* set D of a graph G is a set of vertices such that every vertex outside D is adjacent to a vertex of D. The k-Dominating Set problem is to decide, given a graph G and a positive integer k, whether G has a dominating set of size k.

k-Dominating set



k-Dominating set is not closed under taking minors. However, it is closed under contraction of edges.

Subexponential algorithms on planar graphs: What is the main idea?

Dynamic programming and Grid Theorem

Meta conditions

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

(B) For every graph $G \in \mathcal{G}$ and given a branch decomposition (T, μ) of G, the value of $P(G)$ can be computed in $f(\mathbf{bw}(T, \mu)) \cdot n^{O(1)}$ steps.

Algorithm

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

(B) For every graph $G \in \mathcal{G}$ and given a branch decomposition (T, μ) of G, the value of $P(G)$ can be computed in $f(\mathbf{bw}(T, \mu)) \cdot n^{O(1)}$ steps.

If $\mathbf{bw}(T, \mu) > \alpha \cdot \sqrt{k}$, then by (A) the answer is clear

Else, by (B), $P(G)$ can be computed in $f(\alpha \cdot \sqrt{k}) \cdot n^{O(1)}$ steps.

When $f(k) = 2^{O(k)}$, the running time is $2^{O(\sqrt{k})} \cdot n^{O(1)}$

Using the hammer:

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

(B) For every graph $G \in \mathcal{G}$ and given a branch decomposition (T, μ) of G, the value of $P(G)$ can be computed in $f(\mathbf{bw}(T, \mu)) \cdot n^{O(1)}$ steps

- How to prove (A)?
- How to do (B)?

Combinatorial bounds:

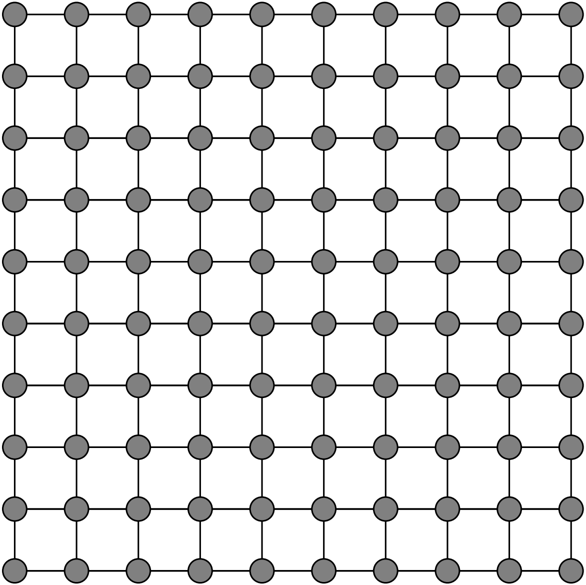
Bidimensionality and excluding a grid as a minor

Reminder

Theorem (Robertson, Seymour & Thomas, 1994)

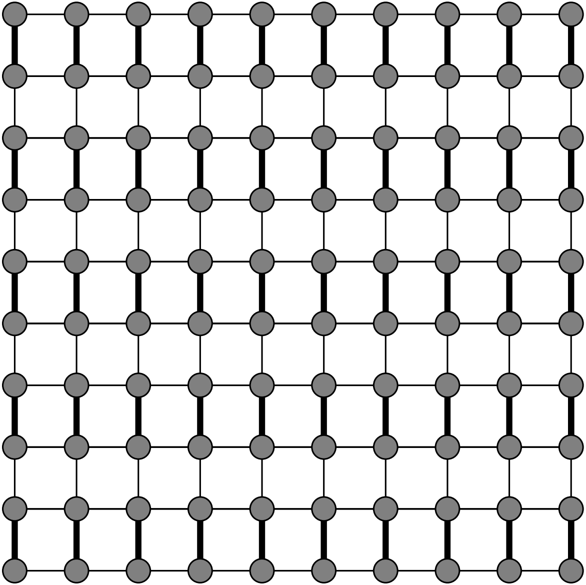
Let $\ell \geq 1$ be an integer. Every planar graph of branchwidth $\geq \ell$ contains an $(\ell/4 \times \ell/4)$-grid as a minor.

PLANAR k-VERTEX COVER



$H_{r,r}$ for $r = 10$

Planar k-Vertex Cover



$\mathbf{vc}(H_{r,r}) \geq \frac{r^2}{2}$

Planar k-Vertex Cover

Let G be a planar graph of

branchwidth $\geq \ell$

PLANAR k-VERTEX COVER

Let G be a planar graph of branchwidth $\geq \ell$ $\Longrightarrow$ G contains an $(\ell/4 \times \ell/4)$-grid H as a minor

Planar k-Vertex Cover

Let G be a planar graph of branchwidth $\geq \ell$ $\Longrightarrow$ G contains an $(\ell/4 \times \ell/4)$-grid H as a minor

The size of any vertex cover of H is at least $\ell^2/32$. Since H is a minor of G, the size of any vertex cover of G is at least $\ell^2/32$.

Planar k-Vertex Cover

Let G be a planar graph of branchwidth $\geq \ell$ $\Longrightarrow$ G contains an $(\ell/4 \times \ell/4)$-grid H as a minor

The size of any vertex cover of H is at least $\ell^2/32$. Since H is a minor of G, the size of any vertex cover of G is at least $\ell^2/32$.

Conclusion: Property (A) holds for $\alpha = 4\sqrt{2}$, i.e.

$\mathbf{bw}(G) \leq 4\sqrt{2}\sqrt{\mathbf{vc}(G)}$.

Planar k-Vertex Cover

Dorn, 2006: given a branch decomposition of G of width ℓ, the minimum vertex cover of G can be computed in time $f(\ell)n = 2^{\frac{\omega}{2}\ell}n$, where ω is the fast matrix multiplication constant.

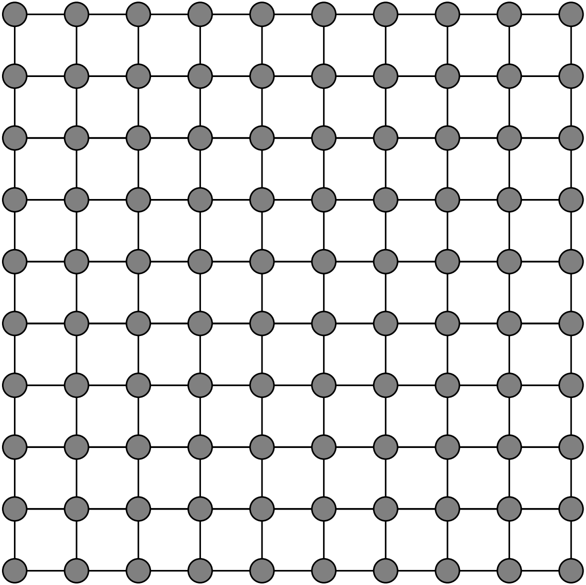
Planar k-Vertex Cover: Putting things together

- Use Seymour-Thomas algorithm to compute a branchwidth of a planar graph G in time $O(n^3)$
- If $\mathbf{bw}(G) \geq \frac{4\sqrt{k}}{\sqrt{2}}$, then G has no vertex cover of size k
- Otherwise, compute vertex cover in time $O(2^{\frac{2\omega\sqrt{k}}{\sqrt{2}}} n) = O(2^{3.56\sqrt{k}} n)$
- Total running time $O(n^3 + 2^{3.56\sqrt{k}} n)$

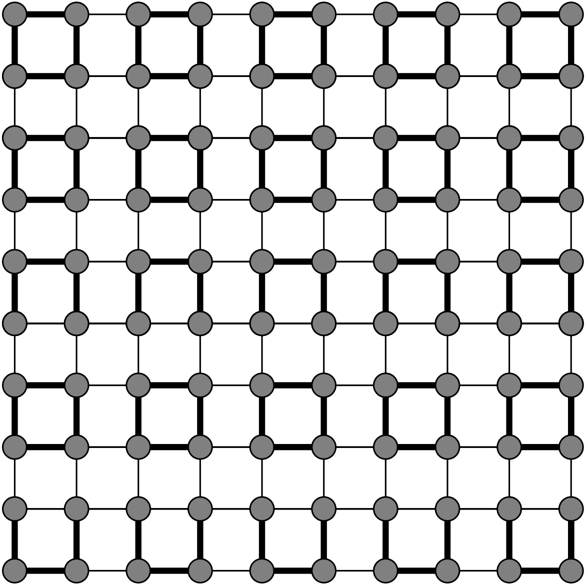
Planar k-Vertex Cover: Kernelization never hurts

- Find a kernel of size $O(k)$ in time $n^{3/2}$ (use Fellows et al. crown decomposition method)
- Use Seymour-Thomas algorithm to compute a branchwidth of the reduced planar graph G in time $O(k^3)$
- If $\mathbf{bw}(G) \geq \frac{4\sqrt{k}}{\sqrt{2}}$, then G has no vertex cover of size k
- Otherwise, compute vertex cover in time $O(2^{\frac{2\omega\sqrt{k}}{\sqrt{2}}} k) = O(2^{3.56\sqrt{k}} k)$
- Total running time $O(n^{3/2} + 2^{3.56\sqrt{k}} k)$

k-Feedback Vertex Set



k-Feedback Vertex Set



$\mathbf{fvc}(H_{r,r}) \geq \frac{r^2}{4}$

k-Feedback Vertex Set

- If $\mathbf{bw}(G) \geq r$, then $G \geq_m H_{\frac{r}{4},\frac{r}{4}}$
- $\mathbf{fvs}$ is minor-closed, therefore $\mathbf{fvs}(G) \geq \mathbf{fvs}(H_{\frac{r}{4},\frac{r}{4}}) \geq \frac{r^2}{64}$

we have that $\mathbf{bw}(G) \leq 8 \cdot \sqrt{\mathbf{fvs}(G)}$

k-Feedback Vertex Set

- If $\mathbf{bw}(G) \geq r$, then $G \geq_m H_{\frac{r}{4},\frac{r}{4}}$
- $\mathbf{fvs}$ is minor-closed, therefore $\mathbf{fvs}(G) \geq \mathbf{fvs}(H_{\frac{r}{4},\frac{r}{4}}) \geq \frac{r^2}{64}$

we have that $\mathbf{bw}(G) \leq 8 \cdot \sqrt{\mathbf{fvs}(G)}$

therefore, for p-Vertex Feedback Set, $f(k) = O(\sqrt{k})$

k-Feedback Vertex Set

- If $\mathbf{bw}(G) \geq r$, then $G \geq_m H_{\frac{r}{4},\frac{r}{4}}$
- $\mathbf{fvs}$ is minor-closed, therefore $\mathbf{fvs}(G) \geq \mathbf{fvs}(H_{\frac{r}{4},\frac{r}{4}}) \geq \frac{r^2}{64}$

we have that $\mathbf{bw}(G) \leq 8 \cdot \sqrt{\mathbf{fvs}(G)}$

therefore, for p-Vertex Feedback Set, $f(k) = O(\sqrt{k})$

Conclusion: Since p-Vertex Feedback Set is "easily" solvable in time $\mathbf{bw}(G)^{\mathbf{bw}(G)}m$, p-Vertex Feedback Set on planar graphs is solvable in time $2^{O(\log k \cdot \sqrt{k})} \cdot O(n)$. (Can be improved to $2^{O(\sqrt{k})} \cdot O(n)$.)

PLANAR k-DOMINATING SET

Can we proceed by the same arguments with PLANAR k-DOMINATING SET?

PLANAR k-DOMINATING SET

Can we proceed by the same arguments with PLANAR k-DOMINATING SET?

Oops! Here is a problem! Dominating set is not minor closed!

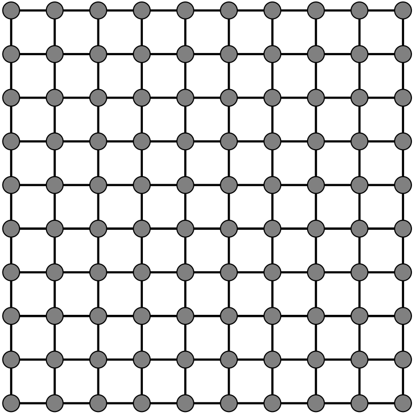
Planar k-Dominating Set

Can we proceed by the same arguments with Planar k-Dominating Set?

Oops! Here is a problem! Dominating set is not minor closed!

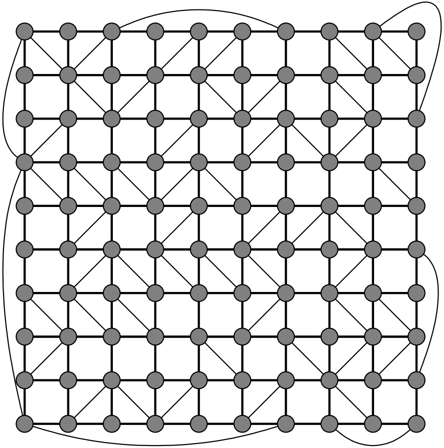
However, dominating set is closed under contraction

PLANAR k-DOMINATING SET



$H_{r,r}$ for $r = 10$

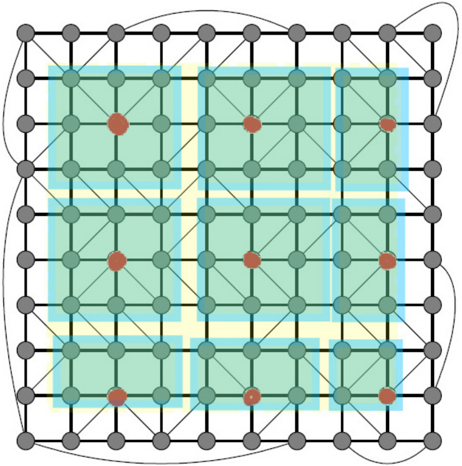
Planar k-Dominating Set



a partial triangulation of

$H_{10,10}$

Planar k-Dominating Set



Every inner vertex of p.t. grid $\tilde{H}_{r,r}$ dominates at most 9 vertices. Thus $ds(\tilde{H}_{r,r}) \geq \frac{(r-2)^2}{9}$.

Planar k-Dominating Set

- By RST-Theorem, a planar graph G of branchwidth $\geq \ell$ can be contracted to a partially triangulated $(\ell/4 \times \ell/4)$-grid
- Since dominating set is closed under contraction, we can make the following

Conclusion: Property (A) holds for $\alpha = 12$, i.e. $\mathbf{bw}(G) \leq 12\sqrt{\mathbf{ds}(G)}$.

Planar k-Dominating Set

- By RST-Theorem, a planar graph G of branchwidth $\geq \ell$ can be contracted to a partially triangulated $(\ell/4 \times \ell/4)$-grid
- Since dominating set is closed under contraction, we conclude that Planar k-Dominating Set also satisfies property (A) with $\alpha = 12$.
- Dorn, 2006, show that for k-Dominating Set in (B), one can choose $f(\ell) = 3^{\frac{\omega}{2}\ell}$, where ω is the fast matrix multiplication constant.

Planar k-Dominating Set

- By RST-Theorem, a planar graph G of branchwidth $\geq \ell$ can be contracted to a partially triangulated $(\ell/4 \times \ell/4)$-grid
- Since dominating set is closed under contraction, we conclude that Planar k-Dominating Set also satisfies property (A) with $\alpha = 12$.
- Dorn, 2006, show that for k-Dominating Set in (B), one can choose $f(\ell) = 3^{\frac{\omega}{2}\ell}$, where ω is the fast matrix multiplication constant.
- Conclusion: Planar k-Dominating Set can be solved in time $O(n^3 + 2^{22.6\sqrt{k}}n)$

Bidimensionality: The main idea

If the graph parameter is closed under taking minors or contractions, the only thing needed for the proof branchwidth/parameter bound is to understand how this parameter behaves on a (partially triangulated) grid.

Bidimensionality: Demaine, FF, Hajiaghayi, Thilikos, 2005

Definition

A parameter P is *minor bidimensional with density* δ if

1. P is closed under taking of minors, and
2. for the $(r \times r)$-grid R, $P(R) = (\delta r)^2 + o((\delta r)^2)$.

Bidimensionality: Demaine, FF, Hajiaghayi, Thilikos, 2005

Definition

A parameter P is called *contraction bidimensional with density* δ if

1. P is closed under contractions,
2. for any partially triangulated $(r \times r)$-grid R,

   $P(R) = (\delta_R r)^2 + o((\delta_R r)^2)$, and
3. δ is the smallest δ_R among all paritally triangulated

   $(r \times r)$-grids.

Bidimensionality

(A) For every graph $G \in \mathcal{G}$, $\mathbf{bw}(G) \leq \alpha \cdot \sqrt{P(G)} + O(1)$

Lemma

If P is a bidimensional parameter with density δ then P satisfies property (A) for $\alpha = 4/\delta$, on planar graphs.

Proof.

Let R be an $(r \times r)$-grid.

$P(R) \geq (\delta_R r)^2$.

If G contains R as a minor, then $\mathbf{bw}(G) \leq 4r \leq 4/\delta\sqrt{P(G)}$. □

Examples of bidimensional problems

Vertex cover

Dominating Set

Independent Set

(k, r)-center

Feedback Vertex Set

Minimum Maximal Matching

Planar Graph TSP

Longest Path ...

How to extend bidimensionality to more general graph classes?

- We need excluding grid theorems (sufficient for minor closed parameters)
- For contraction closed parameters we have to be more careful

Bounded genus graphs: Demaine, FF, Hajiaghayi, Thilikos, 2005

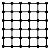
Theorem

If G is a graph of genus at most γ with branchwidth more than r, then G contains a $(r/4(\gamma+1) \times r/4(\gamma+1))$-grid as a minor.

The grid-minor-excluding theorem gives linear bounds for H-minor free graphs:

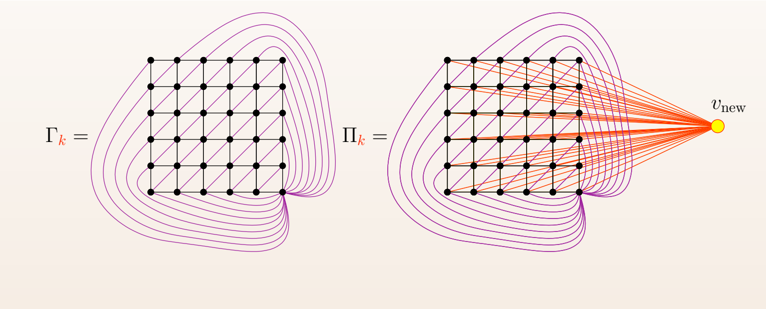
Theorem (Demaine & Hajiaghayi, 2008)

There is a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph G excluding a fixed h-vertex graph H as a minor the following holds:

- *if $\mathbf{bw}(G) \geq \phi(h) \cdot k$ then*  $_k \leq_m G$.

What about contraction-closed parameters?

We define the following two pattern graphs Γ_k and Π_k:

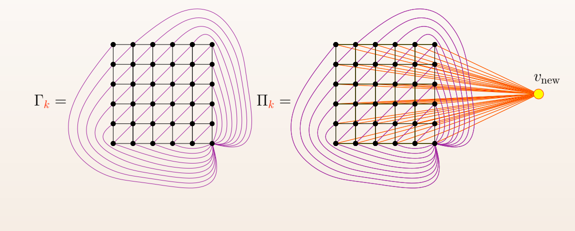


$\Pi_k = \Gamma_k +$ a new vertex v_{new}, connected to all the vertices in $V(\Gamma_k)$.

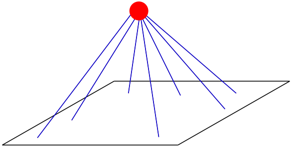
Theorem (FF, Golovach, & Thilikos, 2009)

There is a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ *such that for every graph* G *excluding a fixed* h*-vertex graph* H *as contraction the following holds:*

- *if* $\mathbf{bw}(G) \geq \phi(h) \cdot k$ *then either* $\Gamma_k \leq_c G$*, or* $\Pi_k \leq_c G$*.*



H^* is an *apex graph* if



$\exists v \in V(H^*)$: $H^* - v$ is planar

(*apex graphs* are exactly the minors of Π_k)

Corollary

There is a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ *such that for every graph* G *excluding a fixed* h*-vertex apex graph* H *as contraction the following holds:*

- *if* $\mathbf{bw}(G) \geq \phi(h) \cdot k$ *then* $\Gamma_k \leq_c G$.

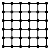
(Redefining contraction bidimensionality

For contraction-closed graph class a contraction-closed parameter

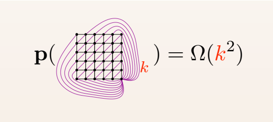
$\mathbf{p}$ is bidimensional if

$$\mathbf{p}(\Gamma_k) = \Omega(k^2).$$

Conclusion

Minor bidimensional: minor- closed and $\mathbf{p}($$_{k}) = \Omega(k^2)$

Contraction-bidimensional: contraction-closed and

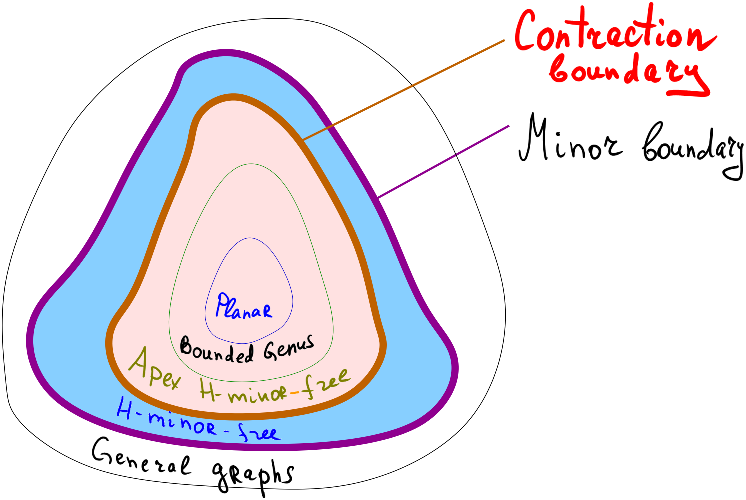


Theorem (Bidimensionality meta-algorithm)

Let $\mathbf{p}$ be a minor (resp. contraction)-bidimensional parameter that is computable in time $2^{O(\mathbf{bw}(G))} \cdot n^{O(1)}$.

Then, deciding $\mathbf{p}(G) \leq k$ for general (resp. apex) minor-free graphs can be done (optimally) in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Limits of the bidimensionality



Remark

Bidimensionality cannot be used to obtain subexponential algorithms for contraction closed parameterized problems on H-minor free graphs.

For some problems, like k-Dominating Set, it is still possible to design subexponential algorithms on H-minor free graphs.

The main idea here is to use decomposition theorem of Robertson-Seymour about decomposing an H-minor free graph into pieces of apex-minor-free graphs, apply bidimensionality for each piece, and do dynamic programming over the whole decomposition.