Strategic Games: a Mini-Course for Computer Scientists

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Basic Concepts

Overview

- Best response.
- Nash equilibrium.
- Pareto efficient outcomes.
- Social welfare.
- Social optima.
- Examples.

Strategic Games: Definition

Strategic game for $n \geq 2$ players:

- (possibly infinite) set S_i of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$,

for each player i.

Basic assumptions:

- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others' rationality.

Three Examples

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Main Concepts

- Notation: $s_i, s'_i \in S_i$, $s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n$.
- s_i is a best response to s_{-i} if

$$\forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- s is a Nash equilibrium if $\forall i \;\; s_i$ is a best response to s_{-i}:

$$\forall i \in \{1, \ldots, n\} \; \forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- s is Pareto efficient if for no s'

$$\forall i \in \{1, \ldots, n\} \; p_i(s') \geq p_i(s),$$
$$\exists i \in \{1, \ldots, n\} \; p_i(s') > p_i(s).$$

- Social welfare of s: $\sum_{j=1}^{n} p_j(s)$.
- s is a social optimum if $\sum_{j=1}^{n} p_j(s)$ is maximal.

Nash Equlibrium

Prisoner's Dilemma: 1 Nash equilibrium

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The Battle of the Sexes: 2 Nash equilibria

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies: no Nash equlibrium

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

- 1 Nash equilibrium: (D, D),
- 3 Pareto efficient outcomes: (C, C), (C, D),(D, C),
- 1 social optimum: (C, C).

Prisoner's Dilemma for n Players

- $n > 1$ players,
- two strategies:
  1 (formerly C),
  0 (formerly D).

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

- For $n = 2$ we get the original Prisoner's Dilemma game.
- Let **1** $= (1, \ldots, 1)$ and **0** $= (0, \ldots, 0)$.
- **0** is the unique Nash equilibrium, with social welfare n.
- Social optimum: **1**, with social welfare $2n(n-1)$.

Tragedy of the Commons

- Common resources: goods that are are not *excludable* (people cannot be prevented from using them) but are *rival* (one person's use of them diminishes another person's enjoyment of it).
- Examples: congested toll-free roads, fish in the ocean, the environment, . . .,
- Problem: Overuse of such common resources leads to their destruction.
- This phenomenon is called the tragedy of the commons (Hardin '81).

Tragedy of the Commons I

(Gardner '95)

- $n > 1$ players,
- two strategies:
  1 (use the resource),
  0 (don't use),
- payoff function:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0 \\ F(m)/m & \text{otherwise} \end{cases}$$

where $m = \sum_{j=1}^{n} s_j$ and

$$F(m) := 1.1m - 0.1m^2.$$

Tragedy of the Commons I, ctd

- payoff function:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0 \\ F(m)/m & \text{otherwise} \end{cases}$$

  where $m = \sum_{j=1}^{n} s_j$ and $F(m) := 1.1m - 0.1m^2$.
- Note: $F(m)/m$ is strictly decreasing, $F(9)/9 = 0.2$, $F(10)/10 = 0.1$, $F(11)/11 = 0$.
- Nash equilibria:
  $n < 10$: all players use the resource,
  $n \geq 10$: 9 or 10 players use the resource,
- Social optimum: 5 players use the resource.

Tragedy of the Commons II

(Osborne '04)

- $n > 1$ players,
- strategies: $[0, 1]$,
- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Tragedy of the Commons II, ctd

- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- 'Best' Nash equilibrium:
  when each $s_i = \frac{1}{n+1}$,
  with social welfare $\frac{n}{(n+1)^2}$ and $\sum_{j=1}^{n} s_j = \frac{n}{n+1}$.
- Social optimum, when $\sum_{j=1}^{n} s_j = \frac{1}{2}$,
  with social welfare $\frac{1}{4}$.
- For all $n > 1$, $\frac{n}{(n+1)^2} < \frac{1}{4}$.
- $\lim_{n} \rightarrow \infty \frac{n}{(n+1)^2} = 0$ and $\lim_{n} \rightarrow \infty \frac{n}{n+1} = 1$.

More on Nash Equilibria

Overview

- Best response dynamics.
- Potential games.
- Congestion games.
- Examples.
- Price of Stability.
- Mixed strategies.
- Nash Theorem.

Best Response Dynamics

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- An algorithm to find a Nash equilibrium:

choose $s \in S_1 \times \ldots \times S_n$;
while s is not a NE **do**
  choose $i \in \{1, \ldots, n\}$ such that
    s_i is not a best response to s_{-i};
  $s_i :=$ a best response to s_{-i}
od

- Example: the Battle of the Sexes game.

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Best Response Dynamics, ctd

Best response dynamics may miss a Nash equilibrium.

Example (Shoham and Leyton-Brown '09)

	H	T	E
H	$1, -1$	$-1, \quad 1$	$-1, -1$
T	$-1, \quad 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

Potential Games

(Monderer and Shapley '96)

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- Function $P : S_1 \times \ldots S_n \rightarrow \mathbb{R}$ is a potential function for G if

$$\forall i \in \{1, \ldots, n\} \; \forall s_{-i} \in S_{-i} \; \forall s_i, s'_i \in S_i$$
$$p_i(s_i, s_{-i}) - p_i(s'_i, s_{-i}) = P(s_i, s_{-i}) - P(s'_i, s_{-i}).$$

- Intuition: P tracks the changes in the payoff when some player deviates.
- Potential game: a game that has a potential function.

Example

- Prisoner's dilemma for n players.

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

- For $i = 1, 2$

$$p_i(0, s_{-i}) - p_i(1, s_{-i}) = 1.$$

- So $P(s) := -\sum_{j=1}^{n} s_j$ is a potential function.

Non-examples

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Each potential function P would have to satisfy

- $P(F, F) - P(B, F) = 2$,
- $P(F, F) - P(F, B) = 1$,
- $P(B, B) - P(F, B) = -1$,
- $P(B, B) - P(B, F) = -2$.

Matching Pennies See the next slide.

Potential Games, ctd

Theorem (Monderer and Shapley '96)
Every finite potential game has a Nash equilibrium.

Proof 1.

- The games $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $(S_1, \ldots, S_n, P, \ldots, P)$ have the same set of Nash equilibria.
- Take s for which P reaches maximum. Then s is a Nash equilibrium of $(S_1, \ldots, S_n, P, \ldots, P)$.

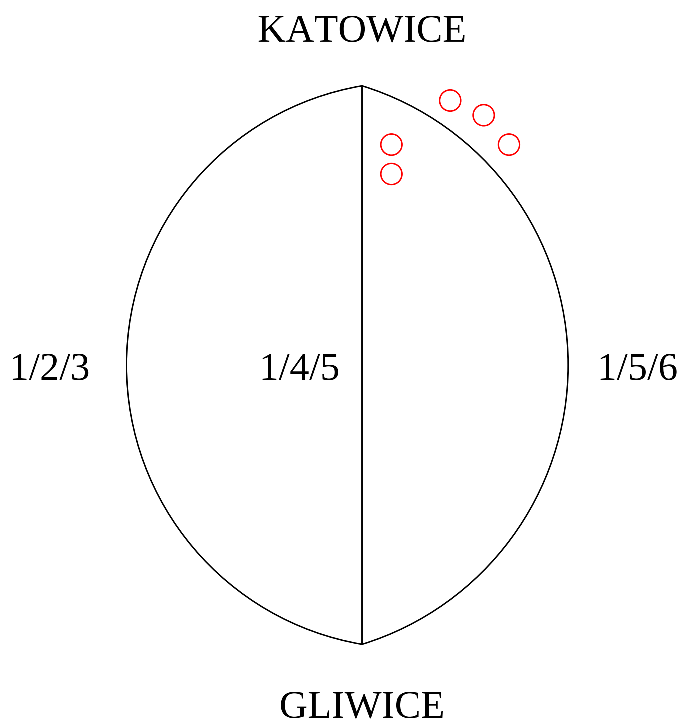
Proof 2.
For finite potential games the best response dynamics terminates.

Congestion Games

- $n > 1$ players,
- set M of facilities (road segments, primary production factors, $\ldots$),
- each strategy is a non-empty subset of M,
- each player has a possibly different set of strategies,
- $c_j : \{1, \ldots, n\} \to \mathbb{R}$ is the cost function for using $j \in M$,
- $c_j(k)$ is the cost to each user of facility j when there are k users of j,
- $u_j(s) := |\{r \in \{1, \ldots, n\} \mid j \in s_r\}|$ is the number of users of facility j given s,
- $c_i(s) := \sum_{j \in s_i} c_j(u_j(s))$,
- (We use here cost functions c_i instead of payoff functions p_i.)

Przykład

- 5 kierowców.
- Każdy kierowca wybiera drogę z Katowic do Gliwic,
- Więcej kierowców wybiera tę samą drogę: większe opóźnienia.

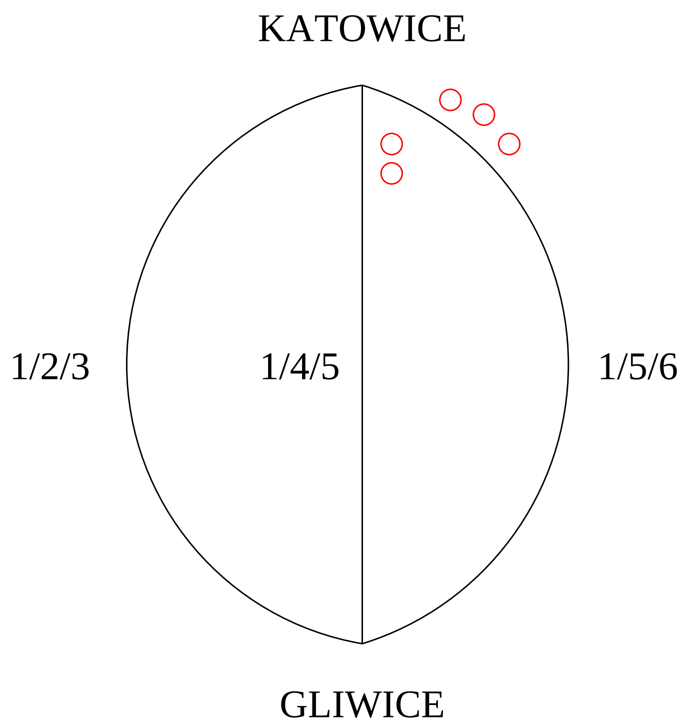


Przykład jako ’Congestion Game’

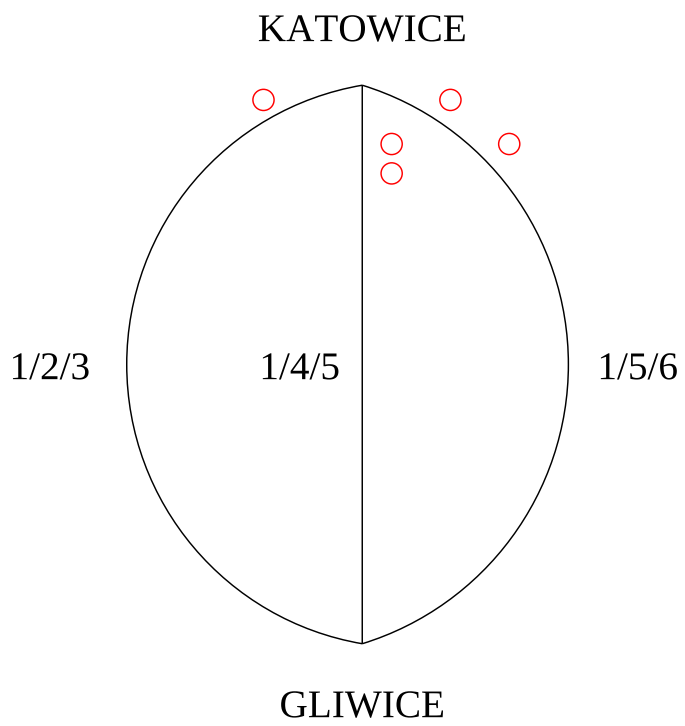
- 5 players,
- 3 facilities (roads),
- each strategy: (a singleton set consisting of) a road,
- cost function:

$$c_i(s) := \begin{cases} 1 & \text{if } s_i = 1 \text{and } |j \mid s_j = 1| = 1 \\ 2 & \text{if } s_i = 1 \text{and } |j \mid s_j = 1| = 2 \\ 3 & \text{if } s_i = 1 \text{and } |j \mid s_j = 1| \geq 3 \\ 1 & \text{if } s_i = 2 \text{and } |j \mid s_j = 2| = 1 \\ \ldots & \\ 6 & \text{if } s_i = 3 \text{and } |j \mid s_j = 3| \geq 3 \end{cases}$$

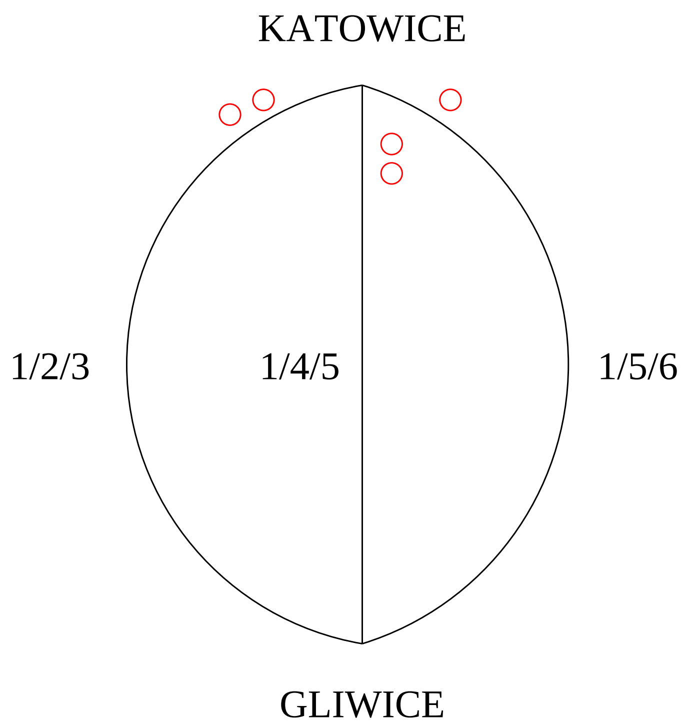
Możliwy Rozwój Wydarzeń (1)



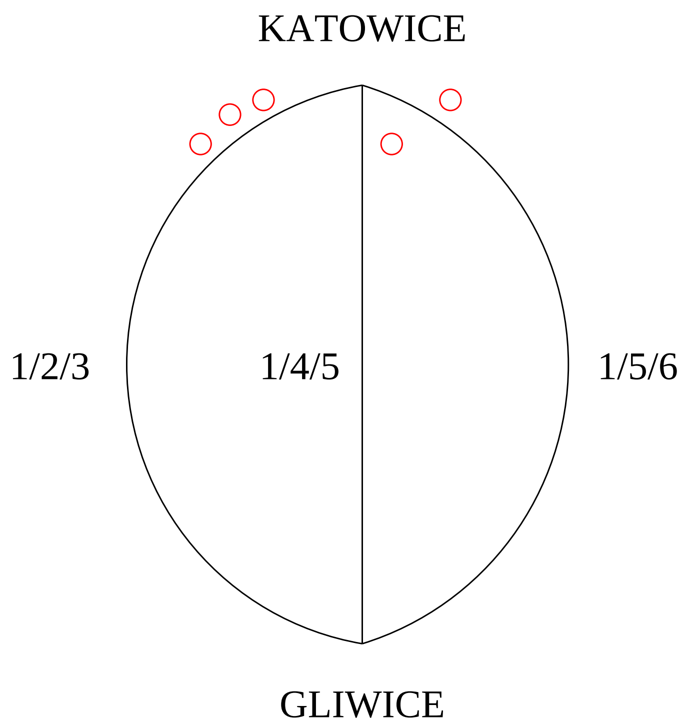
Możliwy Rozwój Wydarzeń (2)



Możliwy Rozwój Wydarzeń (3)



Możliwy Rozwój Wydarzeń (4)



Osiągneliśmy równowagę Nasha, przy użyciu dynamiki najlepszej odpowiedzi (best response dynamics).

Congestion Games, ctd

Theorem (Rosenthal, '73)
Every congestion game is a potential game.

Proof.

$$P(s) := \sum_{j \in s_1 \cup \ldots \cup s_n} \sum_{k=1}^{u_j(s)} c_j(k),$$

where (recall) $u_j(s) = |\{r \in \{1, \ldots, n\} \mid j \in s_r\}|$,

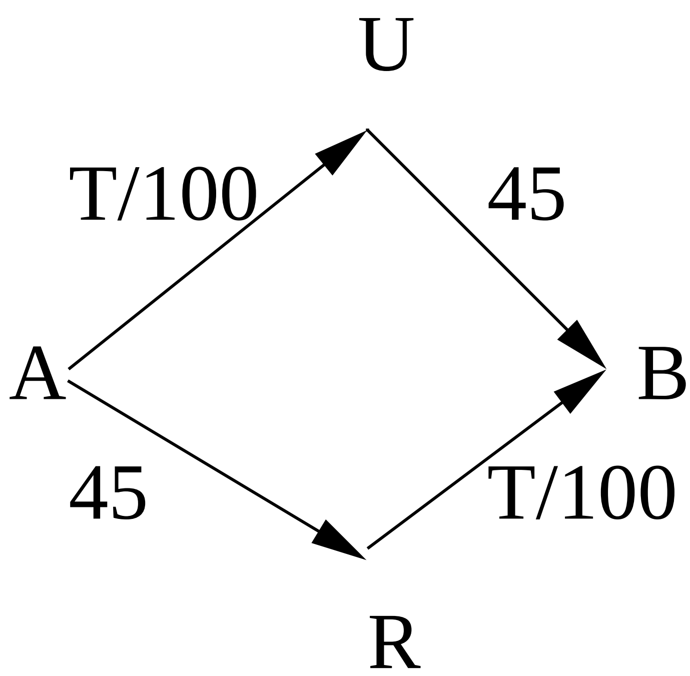
is a potential function.

Conclusion Every congestion game has a Nash equilibrium.

Inny Przykład

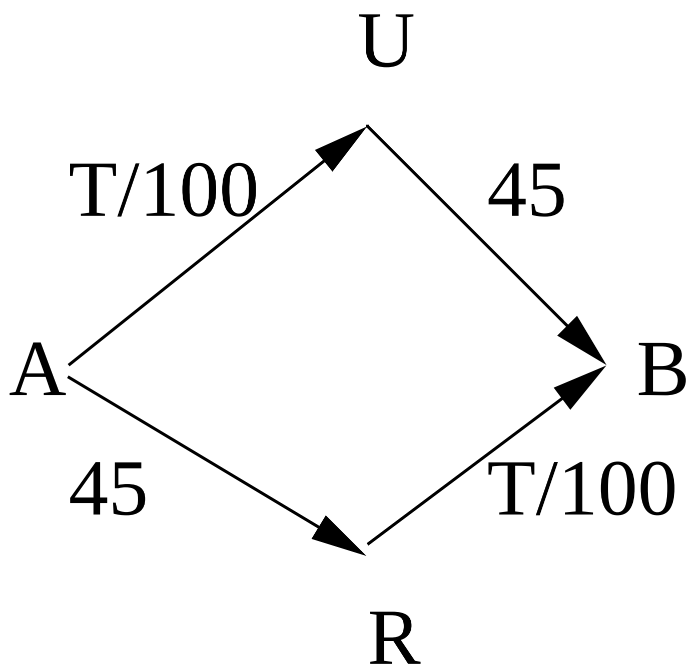
Założenia:

- 4000 kierowców jedzie z A do B.
- Każdy kierowca ma 2 możliwości (strategie).



Problem: Znajdź równowagę Nasha (T = liczba kierowców).

Równowaga Nasha

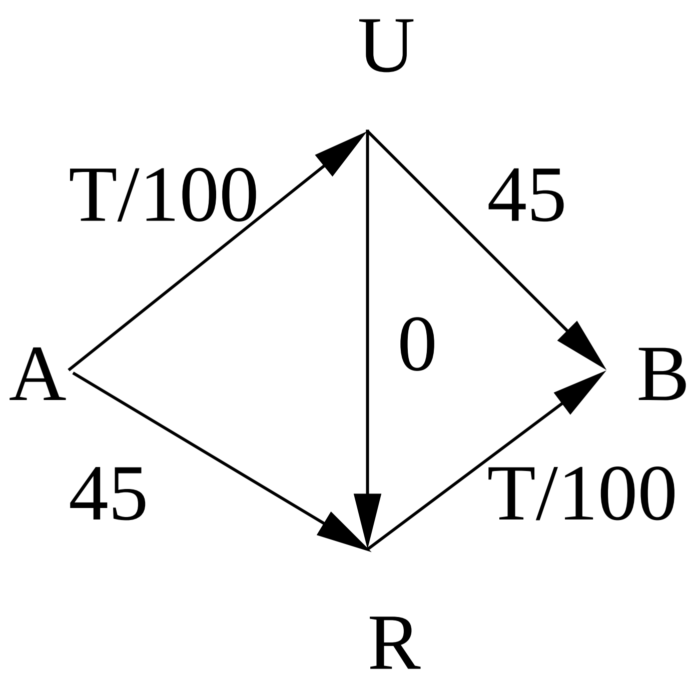


Odpowiedź: 2000/2000.

Czas jazdy: 2000/100 + 45 = 45 + 2000/100 = 65.

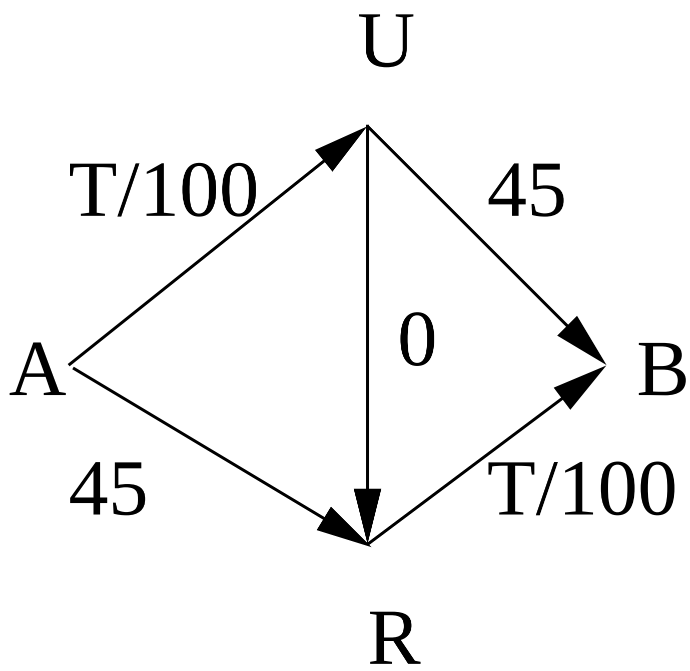
Paradoks Braessa

- Dodaj szybką drogę z U do R.
- Każdy kierowca ma teraz 3 możliwości (strategie):
  A - U - B,
  A - R - B,
  A - U - R - B.



Problem: Znajdź równowagę Nasha.

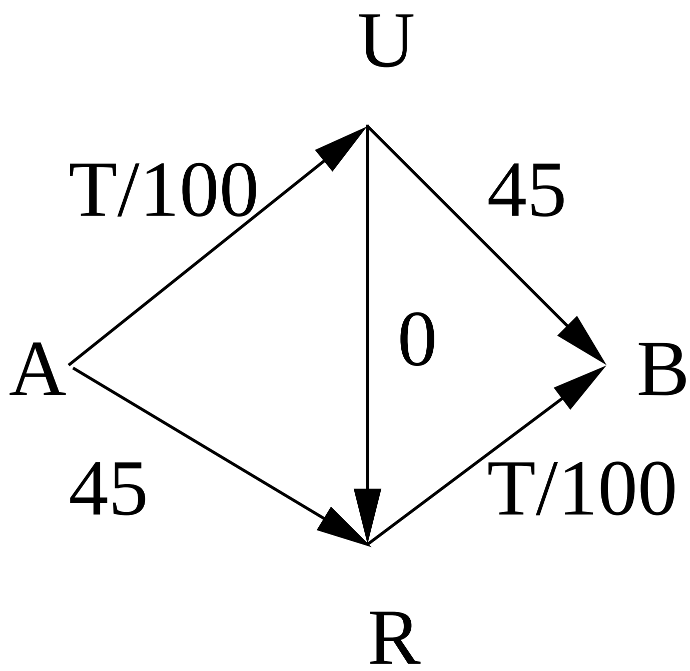
Równowaga Nasha



Odpowiedź: Każdy kierowca wybierze drogę A - U - R - B.

Dlaczego?: Droga A - U - R - B jest zawsze najlepszą odpowiedzią (best response).

Mała komplikacja



Czas jazdy: 4000/100 + 4000/100 = 80!

Czy to się zdarza?

z Wikipedii (‘Braess Paradox’):

- In Seoul, South Korea, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project.
- In Stuttgart, Germany after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.
- In 1990 the closing of 42nd street in New York City reduced the amount of congestion in the area.
- In 2008 Youn, Gastner and Jeong demonstrated specific routes in Boston, New York City and London where this might actually occur and pointed out roads that could be closed to reduce predicted travel times.

Cena Stabilności (Price of Stability)

Definicja

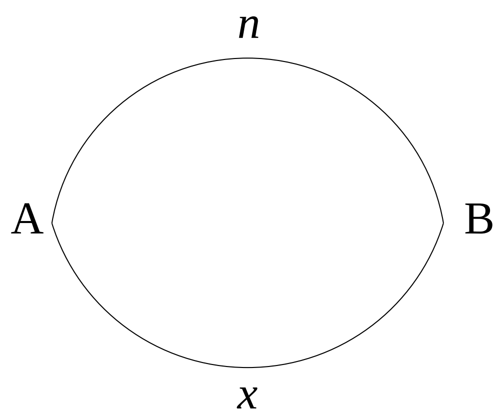
CS: $\dfrac{\text{koszty społeczne najlepszej równowagi Nasha}}{\text{społeczne optimum}}$

In English:

PoS: $\dfrac{\text{social welfare of the best Nash equilibrium}}{\text{social welfare of the social optimum}}$

Pytanie: Ile wynosi CS dla ‘congestion games’?

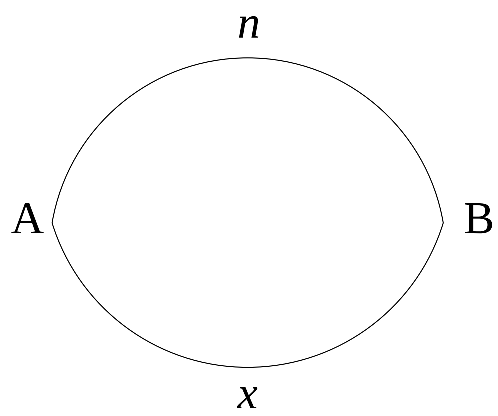
Przykład



n - (parzysta) ilość graczy.
x - ilość kierowców na dolnej drodze.

- Dwie równowagi Nasha
  $1/(n-1)$, z kosztem społecznym $n+(n-1)^2$.
  $0/n$, z kosztem społecznym n^2.
- Społeczne optimum
  Weźmy $f(x) = x \cdot x + (n-x) \cdot n = x^2 - n \cdot x + n^2$.
  Chcemy znaleźć minimum f.
  $f'(x) = 2x - n$, więc $f'(x) = 0$ jeśli $x = \frac{n}{2}$.

Przykład



- Najlepsza równowaga Nasha
  $1/(n-1)$, z kosztem społecznym $n+(n-1)^2$.
- Społeczne optimum
  $f(x) = x^2 - n \cdot x + n^2$.
  Społeczne optimum = $f(\frac{n}{2}) = \frac{3}{4}n^2$.
- CS = $(n+(n-1)^2)/\frac{3}{4}n^2 = \frac{4}{3}\frac{n+(n-1)^2}{n^2}$.
- $\lim_{n\to\infty}$ CS = $\frac{4}{3}$.

Cena Stabilności

- Twierdzenie (Roughgarden i Tárdos, 2002)
  Załóżmy, że funkcje opóźnień (n.p. $T/100$) są liniowe.
  Wówczas CS dla 'congestion games' jest $\leq \frac{4}{3}$.
- Dobrą równowagę Nasha można osiągnąć przy użyciu dynamiki najlepszej odpowiedzi (best response dynamics).
- Niestety: może zabrać wykładniczo długo zanim osiągniemy równowagę.

Mixed Extension of a Finite Game

- Probability distribution over a finite non-empty set A:

  $$\pi : A \rightarrow [0, 1]$$

  such that $\sum_{a \in A} \pi(a) = 1$.
- Notation: ΔA.

Fix a finite strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.

- Mixed strategy of player i in G: $m_i \in \Delta S_i$.
- Joint mixed strategy: $m = (m_1, \ldots, m_n)$.

Mixed Extension of a Finite Game (2)

- Mixed extension of G:

$$(\Delta S_1, \ldots, \Delta S_n, p_1, \ldots, p_n),$$

where

$$m(s) := m_1(s_1) \cdot \ldots \cdot m_n(s_n)$$

and

$$p_i(m) := \sum_{s \in S} m(s) \cdot p_i(s).$$

- Theorem (Nash '50) Every mixed extension of a finite strategic game has a Nash equilibrium.

Kakutani's Fixed Point Theorem

Theorem (Kakutani '41)
Suppose A is a compact and convex subset of $\mathbb{R}^n$ and

$$\Phi : A \rightarrow \mathcal{P}(A)$$

is such that

- $\Phi(x)$ is non-empty and convex for all $x \in A$,
- for all sequences (x_i, y_i) converging to (x, y)

  $y_i \in \Phi(x_i)$ for all $i \geq 0$,

  implies that

  $y \in \Phi(x)$.

Then $x^* \in A$ exists such that $x^* \in \Phi(x^*)$.

Proof of Nash Theorem

Fix $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Define

$$best_i : \Pi_{j \neq i} \Delta S_j \rightarrow \mathcal{P}(\Delta S_i)$$

by

$$best_i(m_{-i}) := \{m'_i \in \Delta S_i \mid p_i(m'_i, m_{-i}) \text{ attains the maximum}\}.$$

Then define

$$best : \Delta S_1 \times \ldots \Delta S_n \rightarrow \mathcal{P}(\Delta S_1 \times \ldots \times \Delta S_n)$$

by

$$best(m) := best_1(m_{-1}) \times \ldots \times best_1(m_{-n}).$$

Note m is a Nash equilibrium iff $m \in best(m)$.
$best(\cdot)$ satisfies the conditions of Kakutani's Theorem.

Comments

- First special case of Nash theorem: Cournot (1838).
- Nash theorem generalizes von Neumann's Minimax Theorem ('28).
- An alternative proof (also by Nash) uses Brouwer's Fixed Point Theorem.
- Search for conditions ensuring existence of Nash equilibrium.

2 Examples

Matching Pennies

	H	T
H	$1, -1$	$-1, \ 1$
T	$-1, \ 1$	$1, -1$

- $(\frac{1}{2} \cdot H + \frac{1}{2} \cdot T, \frac{1}{2} \cdot H + \frac{1}{2} \cdot T)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: 0.

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

- $(2/3 \cdot F + 1/3 \cdot B, 1/3 \cdot F + 1/3 \cdot B)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: $2/3$.

References

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- Modeling Network Traffic using Game Theory. (Chapter 8 from D. Easley and J. Kleinberg, *Networks, Crowds, and Markets: Reasoning About a Highly Connected World.* Cambridge University Press, 2010. To appear.) `www.cs.cornell.edu/home/kleinber/networks-book`

Mechanism Design

Overview

- Decision problems.
- Direct mechanisms.
- Groves mechanisms.
- Examples.
- Optimality results.

Intelligent Design

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Economics focus

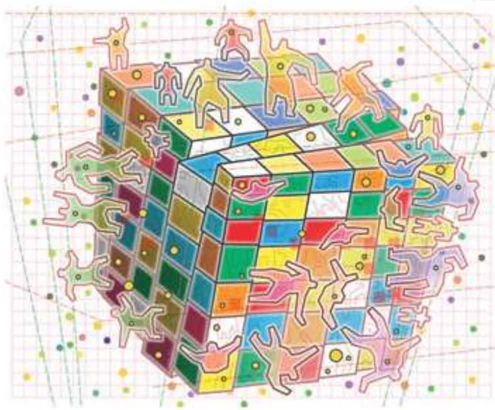
Intelligent design

Oct 18th 2007

From *The Economist* print edition

A theory of an intelligently guided invisible hand wins the Nobel prize

Jac



"WHAT on earth is mechanism design?" was the typical reaction to this year's Nobel prize in economics, announced on October 15th. In this era of "Freakonomics", in which everyone is discovering their inner economist, economics has become unexpectedly sexy. So what possessed the Nobel committee to honour a subject that sounds so thoroughly dismal? Why didn't they follow the lead of the peace-prize judges, who know not to let technicalities about being true to the meaning of the award get in the way of good headlines?

In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like. The word "mechanism" refers to the institutions and the rules of the game that govern our economic activities, which can range from a Ministry of Planning in a command economy to the internal organisation of a company to trading in a market.

Intelligent Design

A theory of an intelligently guided invisible hand wins the Nobel prize

WHAT on earth is mechanism design? was the typical reaction to this year's Nobel prize in economics, announced on October 15th.
[...]
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(The Economist, Oct. 18th, 2007)

Decision Problems

Decision problem for n players:

- set D of decisions,
- for each player i a set of (private) types Θ_i
- and a utility function

$$v_i : D \times \Theta_i \rightarrow \mathcal{R}.$$

- Intuitions
  - Type is some private information known only to the player (e.g., player's valuation of the item for sale),
  - $v_i(d, \theta_i)$ represents the benefit to player i of type θ_i from the decision $d \in D$.
- Assume the individual types are $\theta_1, \ldots, \theta_n$. Then $\sum_{i=1}^{n} v_i(d, \theta_i)$ is the social welfare from $d \in D$.

Decision Rules

- Decision rule is a function

$$f : \Theta_1 \times \ldots \times \Theta_n \to D.$$

- Decision rule f is efficient if

$$\sum_{i=1}^{n} v_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(d, \theta_i)$$

for all $\theta \in \Theta$ and $d \in D$.

- Intuition f is efficient if it always maximizes the social welfare.

Set up

- Each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ_i',
- the central authority computes decision

$$d := f(\theta_1', \ldots, \theta_n'),$$

and communicates it to each player i.

Basic problem How to ensure that $\theta_i' = \theta_i$.

Example 1: Sealed-Bid Auction

Set up There is a single object for sale. Each player is a buyer. The decision is taken by means of a sealed-bid auction. The object is sold to the highest bidder.

- $D = \{1, \ldots, n\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := \begin{cases} \theta_i & \text{if } d = i \\ 0 & \text{otherwise} \end{cases}$
- Let argsmax $\theta := \mu i(\theta_i = \max_{j \in \{1,\ldots,n\}} \theta_j)$.
- $f(\theta) :=$ argsmax θ.
- Note f is efficient.
- Payments will be treated later.

Example 2: Public Project Problem

```
Each person is asked to report his or her
willingness to pay for the project, and
the project is undertaken if and only if
the aggregate reported willingness to pay
exceeds the cost of the project.
```

(15 October 2007, The Royal Swedish Academy of Sciences, Press Release, Scientific Background)

Public Project Problem, formally

- c: cost of the public project (e.g., building a bridge),
- $D = \{0, 1\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := d(\theta_i - \frac{c}{n})$,
- $f(\theta) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$
- Note f is efficient.

Ex. 3: Reversed Sealed-bid Auction

Set up Each player offers the same service. The decision is taken by means of a sealed-bid auction. The service is purchased from the lowest bidder.

- $D = \{1, \ldots, n\}$,
- each Θ_i is $\mathbb{R}_-$;
  $-\theta_i$ is the price player i offers,
- $v_i(d, \theta_i) := \begin{cases} \theta_i & \text{if } d = i \\ 0 & \text{otherwise} \end{cases}$
- $f(\theta) :=$ argsmax θ.

Example $f(-8, -5, -4, -6) = 3$. That is, given the offers $8, 5, 4, 6$, the service is bought from player 3.

xample 4: Buying a Path in a Network

Set up Given a graph $G := (V, E)$.

- Each edge $e \in E$ is owned by player e.
- Two distinguished vertices $s, t \in V$.
- Each player e submits the cost θ_e of using the edge e.
- The central authority selects the shortest $s - t$ path in G.

- $D = \{p \mid p$ is a $s - t$ path in $G\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(p, \theta_i) := \begin{cases} -\theta_i & \text{if } i \in p \\ 0 & \text{otherwise} \end{cases}$
- $f(\theta) := p$, where p is the shortest $s - t$ path in G.

Manipulations

Example An optimal strategy for player i in public project problem:

- if $\theta_i \geq \frac{c}{n}$ submit $\theta'_i = c$.
- if $\theta_i < \frac{c}{n}$ submit $\theta'_i = 0$.

For example, assume $c = 30$.

player	type
A	6
B	7
C	25

Players A and B should submit 0. Player c should submit 30.

Revised Set-up: Direct Mechanisms

- Each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ'_i,
- the central authority computes decision

  $$d := f(\theta'_1, \ldots, \theta'_n),$$

  and taxes

  $$(t_1, \ldots, t_n) := g(\theta'_1, \ldots, \theta'_n) \in \mathbb{R}^n,$$

  and communicates to each player i both d and t_i.
- final utility function for player i:

  $$u_i(d, \theta_i) := v_i(d, \theta_i) + t_i.$$

Direct Mechanisms, ctd

- Direct mechanism (f,t) is incentive compatible if for all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta'_i \in \Theta_i$

$$u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i).$$

- Intuition Submitting false type (so $\theta'_i \neq \theta_i$) does not pay off.
- Direct mechanism (f,t) is feasible if $\sum_{i=1}^{n} t_i(\theta) \leq 0$ for all θ.
- Intuition External financing is never needed.

Groves Mechanisms

- $t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) + h_i(\theta_{-i})$, where

  $h_i : \Theta_{-i} \rightarrow \mathbb{R}$ is an arbitrary function.

- Note

  $$u_i((f,t)(\theta), \theta_i) = \sum_{j=1}^{n} v_j(f(\theta), \theta_j) + h_i(\theta_{-i}).$$

- Intuitions
  - Player i cannot manipulate the value of $h_i(\theta_{-i})$.
  - Suppose $h_i(\theta_{-i}) = 0$.
    When the individual types are $\theta_1, \ldots, \theta_n$
    $u_i((f,t)(\theta), \theta_i)$ is the social welfare from $f(\theta)$.

Groves Theorem

Theorem (Groves '73)
Suppose f is efficient. Then each Groves mechanism is incentive compatible.

Proof.
For all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta'_i \in \Theta_i$

$$u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) = \sum_{j=1}^{n} v_j(f(\theta_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i})$$

$$(f \text{ is efficient}) \geq \sum_{j=1}^{n} v_j(f(\theta'_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i})$$

$$= u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i).$$

Special Case: Pivotal Mechanism

- $h_i(\theta_{-i}) := -\max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j)$.
- Then

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j) \leq 0.$$

- Note Pivotal mechanism is feasible.

Re: Sealed-Bid Auction

Note In the pivotal mechanism

$$t_i(\theta) = \begin{cases} -\max_{j \neq i} \theta_j & \text{if } i = \text{argsmax}\,\theta. \\ 0 & \text{otherwise} \end{cases}$$

So the pivotal mechanism is Vickrey auction (Vickrey '61):
the winner pays the 2nd highest bid.

Example

player	bid	tax to authority	util.
A	18	0	0
B	24	-21	3
C	21	0	0

Social welfare: 0 + 0 + 3 = 3.

Maximizing Social Welfare

Question: Does Vickrey auction maximize social welfare?

Notation θ^*: the reordering of θ is descending order.

Example For $\theta = (1, 4, 2, 3, 1)$ we have
$\theta_{-2} = (1, 2, 3, 0)$,
$(\theta_{-2})^* = (3, 2, 1, 0)$,
so $(\theta_{-2})^*_2 = 2$.

Intuition $(\theta_{-2})^*_2$ is the second highest bid among other bids.

Bailey-Cavallo Mechanism

(Bailey '97, Cavallo '06)

Assume $n \geq 3$.

$$t_i(\theta) := t_i^p(\theta) + \frac{(\theta_{-i})_2^*}{n}$$

Note Bailey-Cavallo mechanism is a Groves mechanism.

Example

player	bid	tax to authority	util.	why?
A	18	0	7	(= $1/3$ of 21)
B	24	-2	9	(= $24 - 2 - 7 - 6$)
C	21	0	6	(= $1/3$ of 18)

Bailey-Cavallo Mechanism, ctd

Note Bailey-Cavallo mechanism is feasible.

Proof. For all i and θ, $(\theta_{-i})^*_2 \leq \theta^*_2$, so

$$\sum_{i=1}^{n} t_i(\theta) = -\theta^*_2 + \sum_{i=1}^{n} \frac{(\theta_{-i})^*_2}{n} = \sum_{i=1}^{n} \frac{-\theta^*_2 + (\theta_{-i})^*_2}{n} \leq 0.$$

Re: Public Project Problem

Assume the pivotal mechanism.

Examples Suppose $c = 30$ and $n = 3$.

player	type	tax	u_i
A	6	0	-4
B	7	0	-3
C	25	-7	8

Social welfare can be negative.

player	type	tax	u_i
A	4	-5	-5
B	3	-6	-6
C	22	0	0

Formally

Note In the pivotal mechanism

$$t_i(\theta) = \begin{cases} 0 & \text{if } \sum_{j \neq i} \theta_j \geq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ \sum_{j \neq i} \theta_j - \frac{n-1}{n} c & \text{if } \sum_{j \neq i} \theta_j < \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ 0 & \text{if } \sum_{j \neq i} \theta_j \leq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \\ \frac{n-1}{n} c - \sum_{j \neq i} \theta_j & \text{if } \sum_{j \neq i} \theta_j > \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \end{cases}$$

This is the mechanism essentially proposed in Clarke '71).

Optimality Result (1)

Theorem (Apt, Conitzer, Guo and Markakis ’08)
Consider the sealed bid auction.
No tax-based mechanism exists that is

- feasible,
- incentive compatible,
- ‘better’ than Bailey-Cavallo mechanism.

Optimality Result (2)

Theorem (Apt, Conitzer, Guo and Markakis ’08)
Consider the public project problem.
No tax-based mechanism exists that is

- feasible,
- incentive compatible,
- ‘better’ than Clarke’s tax.

Proof Steps

1. Limit attention to Groves mechanisms.
   (B. Holmström, '79)
2. Introduce anonymous Groves mechanisms.
3. Pivotal mechanism t is here anonymous.
4. Each Groves mechanism t entails an anonymous Groves mechanism t'.
5. Lemma
   - If t is feasible, then so is t'.
   - If t is 'better' than t_0, then so is t'.
6. Lemma No feasible anonymous Groves mechanism is 'better' than the pivotal mechanism t.

However . . .

Pivotal mechanism is not optimal in the public project problem

- when the payments per player can differ.

Note: Pivotal mechanism then ceases to be anonymous.

Re: Reversed Sealed-Bid Auction

Take

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{d \in D \setminus \{i\}} \sum_{j \neq i} v_j(d, \theta_j).$$

Note

$$t_i(\theta) = \begin{cases} -\max_{j \neq i} \theta_j & \text{if } i = \text{argsmax}\, \theta. \\ 0 & \text{otherwise} \end{cases}$$

So in this mechanism the winner receives the amount equal to the 2nd lowest offer.

Example Consider $\Theta = (-8, -5, -4, -6)$. The service is bought from player 3 who receives for it 5.

Re: Buying a Path in a Network

(Nisan, Ronen '99)
Take

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{p \in D(G \backslash \{i\})} \sum_{j \neq i} v_j(p, \theta_j).$$

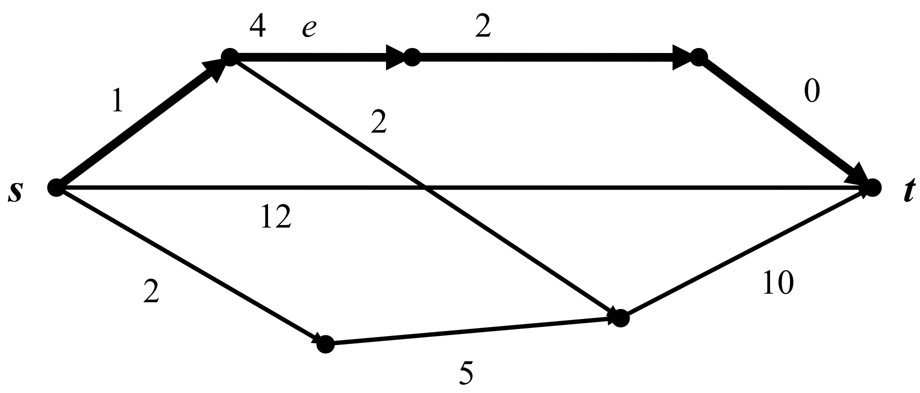
Note

$$t_i(\theta) = \begin{cases} cost(p_2) - cost(p_1 - \{i\}) & \text{if } i \in p_1 \\ 0 & \text{otherwise} \end{cases}$$

where

p_1 is the shortest $s - t$ path in $G(\theta)$,
p_2 is the shortest $s - t$ path in $(G \setminus \{i\})(\theta_{-i})$.

Example



Consider the player owning the edge e.
To compute the payment he receives

- determine the shortest $s - t$ path. Its length is 7. It contains e.
- determine the shortest $s - t$ path that does not include e. Its length is 12.
- So player e receives $12 - (7 - 4) = 9$.
  His final utility is $9 - 4 = 5$.

Pre-Bayesian Games

Pre-Bayesian Games

(Hyafil, Boutilier ’04, Ashlagi, Monderer, Tennenholtz ’06,)

- In a strategic game after each player selected his strategy each player knows all the payoffs (complete information).
- In a pre-Bayesian game after each player selected his strategy each player knows only his payoff (incomplete information).
- This is achieved by introducing (private) types.

Pre-Bayesian Games: Definition

Pre-Bayesian game for $n \geq 2$ players:

- (possibly infinite) set A_i of actions,
- (possibly infinite) set Θ_i of (private) types,
- payoff function $p_i : A_1 \times \ldots \times A_n \times \Theta_i \rightarrow \mathbb{R}$,

for each player i.

Basic assumptions:

- Nature moves first and provides each player i with a θ_i,
- players do not know the types received by other players,
- players choose their actions simultaneously,
- each player is rational (wants to maximize his payoff),
- players have common knowledge of the game and of each others' rationality.

Ex-post Equilibrium

- A strategy for player i:

$$s_i(\cdot) \in A_i^{\Theta_i}.$$

- Joint strategy $s_(\cdot)$ is an ex-post equilibrium if each $s_i(\cdot)$ is a best response to $s_{-i}(\cdot)$:

$$\forall \theta \in \Theta \; \forall i \in \{1, \ldots, n\} \; \forall s'_i(\cdot) \in A_i^{\Theta_i}$$
$$p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq p_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- Note: For each $\theta \in \Theta$ we have one strategic game.
$s(\cdot)$ is an ex-post equilibrium if for each $\theta \in \Theta$ the joint action $(s_1(\theta_1), \ldots, s_n(\theta_n))$ is a Nash equilibrium in the θ-game.

Quiz

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

L (row type U):

	F	B
F	$2, 1$	$2, 0$
B	$0, 1$	$2, 1$

R (row type U):

	F	B
F	$2, 0$	$2, 1$
B	$0, 0$	$2, 1$

L (row type D):

	F	B
F	$3, 1$	$2, 0$
B	$5, 1$	$4, 1$

R (row type D):

	F	B
F	$3, 0$	$2, 1$
B	$5, 0$	$4, 1$

Which strategies form an ex-post equilibrium?

Answer

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

U, L:

	F	B
F	2, 1	2, 0
B	0, 1	2, 1

U, R:

	F	B
F	2, 0	2, 1
B	0, 0	2, 1

D, L:

	F	B
F	3, 1	2, 0
B	5, 1	4, 1

D, R:

	F	B
F	3, 0	2, 1
B	5, 0	4, 1

- Strategies
  $s_1(U) = F, s_1(D) = B$,
  $s_2(L) = F, s_2(R) = B$
  form an ex-post equilibrium.

But . . .

Ex-post equilibrium does not need to exist in mixed extensions of finite pre-Bayesian games.

Example: Mixed extension of the following game.

- $\Theta_1 = \{U, B\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{C, D\}$.

L, U:

	C	D
C	$2, 2$	$0, 0$
D	$3, 0$	$1, 1$

R, U:

	C	D
C	$2, 1$	$0, 0$
D	$3, 0$	$1, 2$

L, B:

	C	D
C	$1, 2$	$3, 0$
D	$0, 0$	$2, 1$

R, B:

	C	D
C	$1, 1$	$3, 0$
D	$0, 0$	$2, 2$

Safety-level Equilibrium

- Strategy $s_i(\cdot)$ for player i is a safety-level best response to $s_{-i}(\cdot)$ if for all strategies $s'_i(\cdot)$ of player i and all $\theta_i \in \Theta_i$

$$\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq \min_{\theta_{-i}\in\Theta_{-i}} p_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- Intuition $\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i)$ is the guaranteed payoff to player i when his type is θ_i and $s(\cdot)$ are the selected strategies.
- Joint strategy $s(\cdot)$ is a safety-level equilibrium if each $s_i(\cdot)$ is a safety-level best response to $s_{-i}(\cdot)$.
- Theorem (Ashlagi, Monderer, Tennenholtz '06)
  Every mixed extension of a finite pre-Bayesian game has a safety-level equilibrium.

Relation to Mechanism Design

- Strategy $s_i(\cdot)$ is dominant if for all $a \in A$ and $\theta_i \in \Theta_i$

$$\forall a \in A \; p_i(s_i(\theta_i), a_{-i}, \theta_i) \geq p_i(a_i, a_{-i}, \theta_i).$$

- A pre-Bayesian game is of a revelation-type if $A_i = \Theta_i$ for all $i \in \{1, \ldots, n\}$.
- So in a revelation-type pre-Bayesian game the strategies of player i are the functions on Θ_i.
- A strategy for player i is called truth-telling if it is the identity function $\pi_i(\cdot)$.

Relation to Mechanism Design, ctd

- Mechanism design (as discussed here) can be viewed as an instance of the revelation-type pre-Bayesian games.
- With each direct mechanism (f, t) we can associate a revelation-type pre-Bayesian game:
  - Each Θ_i as in the mechanism,
  - Each $A_i = \Theta_i$,
  - $p_i(\theta'_i, \theta_{-i}, \theta_i) := u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i)$.
- Note Direct mechanism (f, t) is incentive compatible iff in the associated pre-Bayesian game for each player truth-telling is a dominant strategy.
- Conclusion In the pre-Bayesian game associated with a Groves mechanism, $(\pi_1(\cdot), \ldots, \pi_i(\cdot))$ is a dominant strategy ex-post equilibrium.

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- My lecture notes at `www.cwi.nl/~apt/stra`.

Dziękuję za uwagę